POSITIVITY OF DIVISORS ON BLOWN-UP PROJECTIVE SPACES

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Abstract. We study l-very ample, ample and semi-ample divisors on the blown-up projective space $\mathbb{P}^n$ in a collection of points in general position. We establish Fujita's conjectures for all ample divisors with the number of points bounded above by $2n$ and for an infinite family of ample divisors with an arbitrary number of points.

We construct a log resolution of log pairs on the blown-up space in an arbitrary number of points and we discuss the semi-ampleness of the strict transforms. As an application we prove that the abundance conjecture holds for an infinite family of divisors with an arbitrary number of points.

For $n+2$ points, these strict transforms are F-nef divisors on the moduli space $\overline{\mathcal{M}}_{0,n+3}$ in a Kapranov's model: we show that all of them are nef.

Introduction

Ample line bundles are fundamental objects in Algebraic Geometry. From the geometric perspective, an ample line bundle is one such that some positive multiple of the underlying divisor moves in a linear system that is large enough to give a projective embedding. In numerical terms a divisor is ample if and only if it is in the interior of the real cone generated by nef divisors (Kleiman). Equivalently, a divisor is ample if it intersects positively every closed integral subscheme (Nakai-Moishezon). In cohomological terms, an ample line bundle is one such that a twist of a power by any coherent sheaf is generated by the global sections (Serre). Over the complex numbers, ampleness of line bundles is also equivalent to the existence of a metric with positive curvature (Kodaira).

The very ampleness of divisors on blow-ups of projective spaces and other varieties was studied by several authors, e.g. Beltrametti and Sommese [10], Ballico and Coppens [4], Coppens [20, 21], Harbourne [30]. The notion of l-very ampleness of line bundles was introduced by Beltrametti, Francia and Sommese [8] and l-very ample line bundles on del Pezzo surfaces were classified by Di Rocco [23].

This paper studies ampleness, l-very ampleness and further positivity questions for divisors on blow-ups of projective spaces of arbitrary dimension in points in general position.

The blown-up projective space $\mathbb{P}^n$ in $s \leq n+3$ points is *log Fano*, hence in particular a Mori Dream Space (see for example [2, 17]). It is related to the *moduli*

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space of parabolic vector bundles over $\mathbb{P}^1$ of rank 2 for $s = n+3$ (see [7], [42]) or to the *moduli space of stable rational marked curves* for $s = n+2$

The *effective cone* of divisors on the moduli space of semistable parabolic vector bundles on a rational curve was computed via birational techniques by Moon and Yoo in [40]. Independently, the effective cone as well as the *movable cones* of divisors on the blown-up $\mathbb{P}^n$ in up to $n+3$ points was investigated in [14]; the authors gave equations for the facets of the cones together with a geometric interpretation in terms of joins between linear subspaces spanned by the points and the *secant varieties* of the unique *rational normal curve* of degree n determined by the collection of $n+3$ points. In this paper we use this description to give the *big cone* of divisors for the the blown-up $\mathbb{P}^n$ in at most $n+3$ points, Theorem 3.5.

In a very recent work, Araujo and Massarenti [2] use this description to give an explicit log Fano structure to the blow-up of $\mathbb{P}^n$ in up to $n+3$ points in general position; they do so by studying the blow-up of $\mathbb{P}^n$ along the joins mentioned above.

In [12] the authors analysed the dimension of the space of global sections of divisors on blown-up projective spaces in points in general position. The dimensionality problem is important also in the algebraic framework due to its connections to the *Fröberg-Iarrobino conjectures* that describe the Hilbert series of ideals generated by general powers of linear forms in the polynomial ring with $n+1$ variables (see [12, Section 6] for an account on this). Moreover, in [14] a conjectural formula for the dimension of linear systems on the blow-up of $\mathbb{P}^n$ in up to $n+3$ points, that takes into account the contributions given by the rational normal curve and the joins of its secants with linear subspaces, was given. In this article we pose a series of questions about vanishing cohomology of strict transforms of divisors in the blow-up along the joins of which a positive answer would imply the dimension count, see Subsection 1.4.

In [12, 24] it was shown that each divisor D that is *only linearly obstructed* can be birationally modified by blowing-up its linear base locus and contracting the linear divisorial components, into a divisor $\tilde{D}$, the strict transform of D, such that all higher cohomology groups of $\tilde{D}$ vanish. In this work, we would like to emphasize that the number of global sections of $\tilde{D}$ gives information on the effective cone of divisors while *vanishing theorems* for the higher cohomology groups give information on positivity properties such as the ampleness, the nefness, and the global generation of such divisors.

The first application of the vanishing theorems that will be developed in this paper is the description of *l-very ample divisors*, in particular *globally generated* divisors and *very ample* divisors. This description is contained in Theorem 2.2.

Moreover, in Theorem 4.1, we show that if D is an effective divisor on the blow-up of $\mathbb{P}^n$ in s points in general position, with arbitrary s, that is only linearly obstructed, the strict transforms of D after blowing-up in increasing dimension the linear cycles in the base locus, is base point free (and semi-ample).

Furthermore, in Theorem 5.17, we extend this result to a class of divisors with $s = n+3$ that can have non-linear base locus, by taking the blow-up along the joins between linear cycles and secant varieties to the unique rational normal curve of degree n.

Using these results, we prove a number of conjectures in birational geometry.

First, we establish *Fujita's conjectures* for $\mathbb{P}^n$ blown-up in s points when $s \leq 2n$, Proposition 3.7, and for an infinite family of divisors on $\mathbb{P}^n$ blown-up in arbitrary number of points, with a bound on coefficients, Proposition 3.8.

Moreover, we prove that the *(log) abundance conjecture* holds for an infinite family of log pairs given by effective divisors on the blow-up of $\mathbb{P}^n$ in an arbitrary number of points in general position, Theorem 5.2, in particular providing an explicit construction of *good minimal models*. For a small number of points, $s \leq n+3$, an existence proof of this conjecture follows from work contained in [17, 41] (see also [2]). In this paper we give a constructive proof that extends to an arbitrary number of points, s.

Finally, as explained in [12, Subsection 6.3], the *F-conjecture* predicting the nef cone of $\overline{\mathcal{M}}_{0,n}$ was the original motivation for the study of the vanishing theorems. In this setting, the iterated blow-up of the projective space along all linear cycles in increasing dimension is identified with the moduli space $\overline{\mathcal{M}}_{0,n+3}$ (see [34]). Using vanishing theorems we prove that the conjecture holds for strict transforms of linear systems in $\mathbb{P}^n$ interpolating multiple points in general position, under the blow-up of their linear base locus, Theorem 6.8.

This paper is organized as follows. In Section 1 we introduce the general construction, notation and some preliminary facts. In Subsection 1.4 we pose a number of questions about the cohomology of the strict transform of divisors in the iterated blow-up of $\mathbb{P}^n$ along joins between linear cycles spanned by the points and the secant varieties of the unique rational normal curve of degree n through $n+3$ points.

Section 2 contains one of the main result sof this article, Theorem 2.2, that concerns l-very ampleness of line bundles on blown-up projective spaces in an arbitrary number of points in general position.

In Section 3 we characterize other positivity properties of divisors on blown-up projective spaces at points such as nefness, ampleness, bigness, and we establish Fujita's conjecture for a bounded number of points.

Section 4 contains a description of semi-ample divisors on the iterated blow-up of the projective spaces along linear cycles spanned by subsets of the base points in increasing dimension, Theorem 4.1. Subsection 4.1 contains a complete description of the base locus of linear systems in $\mathbb{P}^n$ interpolating multiple points, Theorem 4.6.

In Section 5 we prove that the abundance conjecture holds for log pairs given by effective divisors on the blow-up of $\mathbb{P}^n$ at s points in general position.

In Section 6, as an application of the results contained in Section 4, we establish the F-conjecture for a particular class of divisors, Theorem 6.8.

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1. Preliminary results and conjectures

Let K be an algebraically closed field of characteristic zero. Let $\mathcal{S} = \{p_1, \dots, p_s\}$ be a collection of s distinct points in $\mathbb{P}^n_K$ and let S be the set of indices parametrizing $\mathcal{S}$, with $|S| = s$.

Let

$$\mathcal{L} := \mathcal{L}_{n,d}(m_1, \dots, m_s) \tag{1.1}$$

denote the linear system of degree-d hypersurfaces of $\mathbb{P}^n$ with multiplicity at least m_i at p_i, for $i = 1, \dots, s$.

1.1. **The blow-up of $\mathbb{P}^n$ in points.** We denote by $X_{s,(0)}$ the blow-up of $\mathbb{P}^n$ in the points of $\mathcal{S}$ and by E_i the exceptional divisor of p_i, for all i. The index (0) indicates that the space $\mathbb{P}^n$ is blown-up in 0-dimensional cycles. The Picard group of $X_{s,(0)}$ is spanned by the class of a general hyperplane, H, and the classes of the exceptional divisors E_i, $i = 1, \dots, s$.

Notation 1.1. Ffix positive integers $d, m_1, \dots, m_s$ and define the following divisor on $X_{s,(0)}$:

$$dH - \sum_{i=1}^{s} m_i E_i \in \operatorname{Pic}\left(X_{s,(0)}\right). \tag{1.2}$$

In this paper we denote by D a *general divisor* in $|dH - \sum_{i=1}^{s} m_i E_i|$. Notice that the global sections of D are in bijection with the elements of the linear system $\mathcal{L}$ defined in (1.1).

Remark 1.2. It is proved in [19, Proposition 2.3] and [15, Lemma 4.2] that for any divisor D of the form (1.2) the general member of $|D|$ has multiplicity equal to m_i at the point p_i. It is important to mention here that even if it is often omitted in the framework of classical interpolation problems in $\mathbb{P}^n$, the generality hypothesis of the divisor D is always assumed.

Theorem 1.3 ([12, Theorem 5.3], [24, Theorem 1.6])**.** *Assume that $\mathcal{S} \subset \mathbb{P}^n$ is a set of points in general position. Let D be as in (1.2). Assume that*

$$\begin{aligned} &0 \le m_i, \ \forall i \in \{1, \dots, s\}, \\ &m_i + m_j \le d + 1, \ \forall i, j \in \{1, \dots, s\}, \ i \ne j, \ (\textit{if } s \ge 2), \\ &\sum_{i=1}^{s} m_i \le nd + \max\{1, s - n - 2\}. \end{aligned} \tag{1.3}$$

Then $h^1(X_{s,(0)}, D) = 0$.

1.2. **Base locus and effectivity of divisors on the blow-up of $\mathbb{P}^n$ in points.** We adopt the same notation as in Subsection 1.1. For any integer $0 \le r \le n-1$ and for any multi-index of cardinality $r+1$,

$$I := \{i_1, \dots, i_{r+1}\} \subseteq \{1, \dots, s\},$$

we define the integer k_I to be the *multiplicity of containment* of the strict transform in $X_{s,(0)}$ of the linear cycle spanned by the points parametrized by I, $L_I \subset \mathbb{P}^n$, in the base locus of D. Notice that under the generality assumption we have $L_I \cong \mathbb{P}^r$.

Lemma 1.4 ([24, Proposition 4.2])**.** *For any effective divisor D as in (1.2), the multiplicity of containment in $\mathrm{Bs}(|D|)$ of the strict transform in $X_{s,(0)}$ of the linear subspace L_I, with $0 \le r \le n-1$, is the integer*

$$k_I = k_I(D) := \max\{0, m_{i_1} + \cdots + m_{i_{r+1}} - rd\}. \tag{1.4}$$

We recall here notations and results introduced in [14]. There exists a unique rational normal curve C of degree n through $n+3$ general points of $\mathbb{P}^n$; this theorem is classically known and its first proof is due to Veronese [45], although it is often attributed to Castelnuovo. Now, let $\sigma_t = \sigma_t(C)$ denote the t-th secant variety of C, namely the Zariski closure of the union of t-secant $(t-1)$-planes. In this notation we have $\sigma_1 = C$. For every $I \subset \{1, \ldots, n+3\}$ with $|I| = r+1$, $-1 \le r \le n$, let

$$\mathrm{J}(L_I, \sigma_t)$$

be the *join* of the linear cycle L_I and σ_t. We use the conventions $|\emptyset| = 0$ and $\sigma_0 = \emptyset$.

The dimension of the variety $\mathrm{J}(L_I, \sigma_t)$ is

$$r = r_{I,\sigma_t} := \dim \mathrm{J}(L_I, \sigma_t) = |I| + 2t - 1. \tag{1.5}$$

Lemma 1.5 ([14, Lemma 4.1])**.** *For any effective divisor D as in (1.2) and for $r_{I,\sigma_t} \le n-1$, the multiplicity of containment in $\mathrm{Bs}(|D|)$ of the strict transform in $X_{s,(0)}$ of the subvariety $\mathrm{J}(L_I, \sigma_t)$ is the integer*

$$k_{I,\sigma_t} = k_{I,\sigma_t}(D) := \max\left\{0, t\sum_{i=1}^{n+3} m_i + \sum_{i\in I} m_i - ((n+1)t + |I| - 1)d\right\}. \tag{1.6}$$

We will now prove a stronger result that will play a crucial role in Section 5, particularly in Proposition 5.22.

Proposition 1.6. *Let D be an effective divisor on $X_{s,(0)}$. Let p be a point on the variety $\mathrm{J}(L_I, \sigma_t)$ that does not lie on any smaller join, namely any $\mathrm{J}(L_{I'}, \sigma_{t'}) \subsetneq \mathrm{J}(L_I, \sigma_t)$, $I' \subset I$, $t' \le t$. Then p is contained in the base locus of the general member of the linear system $|D|$ with multiplicity precisely equal to k_{I,σ_t}.*

Proof. We first prove the claim for $t = 0$ when the join $\mathrm{J}(L_I, \sigma_0)$ is the linear cycle L_I that is in the base locus of the general member of $|D|$ with multiplicity of containment equal to $k_I = k_I(D)$, by Lemma 1.4.

Let us assume, by contradiction, that there is a point p of L_I with multiplicity of containment at least $k_I + 1$. Let $I(r)$ be the largest index set such that $p \in L_{I(r)}$. Lemma 4.2 of [15] implies $r \ge 1$. We introduce the following notation:

$$K_{I(r)} = K_{I(r)}(D) := \sum_{i\in I(r)} m_i - rd.$$

Consider first the case when $K_{I(r)} \ge 1$ and note that $k_{I(r)} = K_{I(r)}$. For $r \ge 2$, let $I(r-2)$ be a subset of $I(r)$, of cardinality equal to $r-1$, and take $H_{I(r-2)}$ a general hyperplane passing through p and all points of $I(r-2)$. For $r = 1$, $H_{I(-1)}$ denotes the hyperplane passing through the point p. For every $r \ge 1$, consider the divisor D', defined as follows, and denote by d' and m'_i its corresponding degree and multiplicities:

$$D' := D + K_{I(r)} H_{I(r-2)}.$$

By assumption, the general member of $|D'|$ contains the point p with multiplicity at least 1. We can compute the multiplicity of containment of $L_{I(r)}$ in the base locus of D':

$$\begin{aligned} K_{I(r)}(D') &= \sum_{i \in I(r)} m_i' - rd' \\ &= \sum_{i \in I(r)} m_i - rd + (r-1)K_{I(r)} - rK_{I(r)} \\ &= K_{I(r)} - K_{I(r)} \\ &= 0. \end{aligned} \tag{1.7}$$

Consider now the case $K_{I(r)} < 0$. Let $H_{I(r)}$ denote a hyperplane containing all points of $I(r)$. Define the divisor

$$D'' := D - K_{I(r)} H_{I(r)}.$$

A similar computation shows the following

$$\begin{aligned} K_{I(r)}(D'') &= \sum_{i \in I(r)} m_i'' - rd'' \\ &= \sum_{i \in I(r)} m_i - rd - (r+1)K_{I(r)} + rK_{I(r)} \\ &= K_{I(r)} - K_{I(r)} \\ &= 0. \end{aligned} \tag{1.8}$$

In the above cases we reduced to the case when D is a divisor with $k_{I(r)} = 0$ whose general member has a base point, p. We now prove by induction on r that this leads to a contradiction.

We discuss separately the case $r = 1$ as the first induction step. The line $L_{I(1)}$ is not contained in the base locus of D by Proposition 1.4. However, the intersection multiplicity between the line $L_{I(1)}$ and D is negative, a contradiction.

In general, we assume that the statement holds for linear cycles of dimension $r-1$ and we prove that it holds for linear cycles of dimension r. The point p can not lie on any smaller linear cycle contained in some $L_{I(r-1)} \subset L_{I(r)}$, by the induction assumption. Therefore, p is a point inside the *interior* of the cycle $L_{I(r)}$, namely $p \in L_{I(r)} \setminus \bigcup_{I(r-1) \subset I(r)} L_{I(r-1)}$. It is easy to see that whenever $k_{I(r)} \geq 0$, then $k_J \geq 0$ for any subset J. This implies that for any subset $I(r-1) \subset I(r)$ of cardinality r, the divisor D contains the linear cycle $L_{I(r-1)}$ in its base locus with multiplicity $k_{I(r-1)} \geq 0$. We consider l a general line in $L_{I(r)}$ passing trough p. We observe that the multiplicity of intersection between l and the divisor D is at most

$$\begin{aligned} d - \sum_{I(r-1) \subset I(r)} k_{I(r-1)} - 1 &= d - \sum_{I(r-1) \subset I(r)} \left(\sum_{i \in I(r-1)} m_i - (r-1)d \right) - 1 \\ &= r \left(\sum_{i \in I(r)} m_i - rd \right) - 1 \\ &= -1. \end{aligned}$$

Since the family of lines passing through the point p covers the linear cycle $L_{I(r)}$ one obtains that $L_{I(r)}$ is in the base locus of the divisor D that is a contradiction with Lemma 1.4, since $k_{I(r)} = 0$.

If $t \geq 1$, the proof follows by a similar argument to the one used [24, Proposition 4.2].

□

We also recall here the classification of effective and movable divisors on the blow-up of $\mathbb{P}^n$ at $s \leq n+3$ points in general position, based on the study of the base locus, Lemma 1.4, that was given in [14]. For $s \leq n+2$ these cones of divisors were originally discussed in [15, Theorem 4.9].

Theorem 1.7 ([14, Theorems 5.1 and 5.3])**.** *We use the same notation as above.*

(1) Assume that $s = n+2$. The divisor D is effective if and only if

$$\begin{aligned} &d \geq 0, \\ &m_i \leq d, \ \forall i \in \{1, \dots, s\}, \\ &\sum_{i \in I} m_i \leq nd, \ \forall I \subset \{1, \dots, s\}, \ |I| = n+1, n+2. \end{aligned} \tag{1.9}$$

Moreover, D is movable if and only if it is effective and

$$\begin{aligned} &m_i \geq 0, \ \forall i \in \{1, \dots, s\}, \\ &\sum_{i \in I} m_i \leq (n-1)d, \ \forall I \subset \{1, \dots, s\}, \ |I| = n. \end{aligned} \tag{1.10}$$

(2) Assume that $s = n+3$. The divisor D is effective if and only if (1.9) and

$$k_{I(n-2t),\sigma_t} \leq 0, \ \forall I(n-2t), \ t \geq 1. \tag{1.11}$$

Moreover, D is movable if and only if it is effective and the following conditions are satisfied: (1.10) and

$$k_{I(n-2t-1),\sigma_t} \leq 0, \ \forall I(n-2t-1), \ t \geq 1. \tag{1.12}$$

are satisfied.

1.3. **The blow-up of $\mathbb{P}^n$ along linear cycles.** For an effective divisor D on $X_{s,(0)}$, we can resolve its linear base locus by constructing a birational morphism $X_{s,(r)} \to X_{s,(0)}$ ([12, Section 4] and [24, Section 2]). For $r \leq \min\{s,n\} - 1$, we denote by $X_{s,(r)}$ the iterated blow-up of $\mathbb{P}^n$ along the strict transform of the linear subspaces L_I of dimension at most r spanned by sets of points $I \subset \mathcal{S}$, $|I| \leq r+1$, with $k_I > 0$, ordered by increasing dimension. We further denote by E_I the (strict transform of) the exceptional divisor in $X_{s,(r)}$ of the linear space L_I, for every such an I.

We point out that the space $X_{s,(r)}$ depends on the divisor D, precisely on the integers k_I, but for the sake of simplicity we omitted this dependency from the notation.

Remark 1.8. Abusing notation, we will denote by L_I the linear subspace of $\mathbb{P}^n =: X_{s,(-1)}$ spanned by the set of points parametrized by I as well as its strict transform in the blown-up spaces $X_{s,(r)}$, for every $r \geq 0$.

(1) For $|I| \geq r+2$, L_I on $X_{s,(r)}$ represents a blown-up projective space of dimension $|I|-1$ along linear cycles of dimension at most r.
(2) If $|I| \leq r+1$, the strict transform of L_I will be the exceptional divisor E_I that is a product of blown-up projective spaces. The full description of this space and its intersection theory is explicitly given in [24, Section 2].

Denote by $D_{(r)}$ the strict transform of the divisor D on $X_{s,(r)}$. One has

$$D_{(r)} := dH - \sum_{\substack{I \subset \{1,\ldots,s\}: \\ 0 \leq |I| \leq r+1}} k_I E_I. \tag{1.13}$$

Remark 1.9. If $r = n-1$, then $D_{(n-1)}$ is the strict transform of $D_{(n-2)}$ via the divisorial contraction $X_{s,(n-2)} \dashrightarrow X_{s,(n-2)}$ obtained by removing $k_{I(n-1)}$ times the strict transform of the hyperplane spanned by the points parametrized by $I(n-1)$, for any $I(n-1) \subset \{1,\ldots,s\}$.

Set $\bar{r}$ to be the maximal dimension of the linear cycles of the base locus of D. We will also use $\tilde{D} := D_{(\bar{r})}$ to denote the strict transform of D under the blow-up of all its linear base locus (including hyperplanes for $\bar{r} = n-1$). To simplify notation we will abbreviate $h^i(X_{s,(r)}, \mathcal{O}_{X_{s,(r)}}(D_{(r)}))$ by $h^i(D_{(r)})$.

Theorem 1.10 ([12, Theorem 4.6])**.** *Let $\mathcal{S}$ be a collection of points of $\mathbb{P}^n$ in general position. Let $D_{(r)}$ be as in (1.13). Assume that*

$$\begin{aligned} &0 \leq m_i \leq d+1, \\ &\sum m_i \leq nd + \max\{1, s-n-2\}. \end{aligned} \tag{1.14}$$

Then $h^i(D_{(r)}) = 0$, for every $i \neq 0, r+1$. Moreover, $h^i(\tilde{D}) = 0$ for every $i \geq 1$.

Theorem 1.10 states that if D is a *special* divisor on $X_{s,(0)}$, i.e. one for which the first cohomology groups does not vanish, and if it satisfies the bounds on the coefficients (1.14), then its strict transform $\tilde{D}$ is no more special, i.e. it has vanishing higher cohomology groups. In [12] the following question was posed, namely whether a similar statement is true for all cycles −linear and not linear− of $\mathrm{Bs}(|D|)$.

Question 1.11 ([12, Question 1.1])**.** *Consider any effective divisor D in the blown-up $\mathbb{P}^n$ at general points. Let $\widetilde{X}$ be the smooth composition of blow-ups of $\mathbb{P}^n$ along the (strict transforms of the) cycles of the base locus of $|D|$, ordered in increasing dimension. Denote by $\widetilde{\mathcal{D}}$ the strict transform of the general divisor of $\mathcal{L}$ in $\widetilde{X}$. Does $h^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\widetilde{\mathcal{D}}))$ vanish for all $i \geq 1$?*

Notice that we adopt the notation $\tilde{D}$ for the strict transform after the blow-up of the linear base locus and $\widetilde{\mathcal{D}}$ for the strict transform after the blow-up of the whole base locus. An affirmative answer to Question 1.11 would imply that $h^0(D) = \chi(Y, \mathcal{O}_Y(\widetilde{\mathcal{D}}))$.

One may ask the following slightly different question.

Question 1.12. *Let X be the blown-up $\mathbb{P}^n$ at general points and D an effective divisor on X. Let $(\widetilde{X}, \widetilde{D})$ be obtained with a log resolution of the pair (X, D). Does $h^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\widetilde{\mathcal{D}}))$ vanish for all $i \geq 1$?*

Notice that the assumption that there exists a log resolution of the pair (X, D) of Question 1.12 is stronger than that of Question 1.11; in fact proving that the strict transform $\widetilde{D}$ is smooth is an extremely difficult task in general.

In Section 4 for $s \leq n+2$ and D effective, or for $s \geq n+3$ and D effective having only linear base locus, Theorem 4.9 establishes that $\widetilde{\mathcal{D}}$ and $\tilde{D}$ coincide, answering positively Question 1.11 and Question 1.12.

1.4. **The blow-up of $\mathbb{P}^n$ along joins of secant varieties of the rational normal curve and linear cycles.** In this section we extend Question 1.11 to the case of effective divisors on the blow-up of $\mathbb{P}^n$ at $n+3$ general points. Take

$$D = dH - \sum_{i=1}^{n+3} m_i E_i \geq 0, \tag{1.15}$$

a divisor on $X_{n+3,(0)}$, the blown-up $\mathbb{P}^n$ at $n+3$ base points, with $d, m_i \geq 0$.

We will iteratively blow-up $X_{n+3,(0)}$ along the subvarieties that are contained in the base locus of D, that are the strict transforms of $\mathrm{J}(L_I, \sigma_t)$, see Lemma 1.5. The pairwise intersections of such subvarieties with some constraints on the index sets parametrizing the vertices, is computed in [2, Proposition 5.6]. For the sake of completeness we state below this result in our notation.

Proposition 1.13 ([2, Proposition 5.6])**.** *Let $I_1, I_2 \subset \{1, \ldots, n+3\}$ be index sets such that $I_1 \cap I_2 = \emptyset$. Let $t_1, t_2 \geq 0$ be integers such that*

$$\begin{aligned} &r_{I_1,\sigma_{t_1}} = r_{I_2,\sigma_{t_2}},\\ &r_{I_i,\sigma_{t_i}} \leq n-1, \ \forall i \in \{1,2\},\\ &2r_{I_i,\sigma_{t_i}} \leq 2n - (|I_1| + |I_2|), \ \forall i \in \{1,2\}. \end{aligned} \tag{1.16}$$

Then

$$\mathrm{J}(L_{I_1}, \sigma_{t_1}) \cap \mathrm{J}(L_{I_2}, \sigma_{t_2}) = \bigcup_J \mathrm{J}(L_J, \sigma_{t_J}),$$

where the union is taken over all subsets $J \subseteq I_1 \cup I_2$ satisfying

$$2|(I_i \cup J) \setminus (I_i \cap J)| = |I_1| + |I_2|, \ \forall i \in \{1,2\},$$

and for every such J, t_J is the integer defined by the following equation

$$2r_{J,\sigma_{t_J}} = 2r_{I_i,\sigma_{t_i}} - (|I_1| + |I_2|).$$

For any effective divisor D, let $\pi^\sigma : Y^\sigma \to X$ be the iterated blow-up of $X := X_{n+3,(0)}$ along (the strict transforms of) all varieties $\mathrm{J}(L_I, \sigma_t)$, $t \geq 0$, $|I| \geq 0$, such that $r_{I,\sigma_t} \leq n-2$ and $k_{I,\sigma_t} > 0$ in increasing dimension, composed with a contraction of the strict transforms of the divisors $\mathrm{J}(L_I, \sigma_t)$ with $r_{I,\sigma_t} = n-1$ and $k_{I,\sigma_t} > 0$. The latter divisors were described in [14, Section 3.2]. The space Y^σ is constructed by Araujo and Massarenti in their recent preprint [2, Section 5] in order to give explicit log Fano structures on $X_{n+3,(0)}$ (see [2, Propositions 5.8,5.11]).

We denote by E_{I,σ_t} the exceptional divisors, for all $t \geq 0$, $|I| \geq 0$, $r_{I,\sigma_t} \leq n-1$. It is immediate to see, using [14, lemma 4.1], that the strict transform on Y^σ of D is given by

$$\tilde{D}^\sigma := dH - \sum_i m_i E_i - \sum_{r=1}^{n-1} \sum_{\substack{I,t:\\ r_{I,\sigma_t}=r}} k_{I,\sigma_t} E_{I,\sigma_t}. \tag{1.17}$$

We stress the fact that the space Y^σ depends on the divisor D.

1.5. **Conjectures on vanishing cohomology on the blow-up along joins of secant varieties of the rational normal curve and linear cycles.** Let D be an effective divisor on $\mathbb{P}^n$ blown-up in $n+3$ general points.

Question 1.14. *Consider the divisor $\tilde{D}_\sigma$ defined in (1.17) as the strict transform of D in Y^σ. Does $h^i(Y^\sigma, \mathcal{O}_{Y^\sigma}(\tilde{D}^\sigma))$ vanish for all $i \geq 1$?*

1.5.1. *Related questions.* Another challenge would be to compute the Euler characteristic of $\tilde{D}^\sigma$. In what follows we would like to propose a candidate for such a number, namely the so called *secant linear virtual dimension* for linear systems of hypersurfaces of $\mathbb{P}^n$ interpolating $n+3$ general points with assigned multiplicity, or equivalently of linear systems $|D|$. This number was introduced in [14].

Definition 1.15 ([14, Definition 6.1])**.** Let D be a divisor on $X_{n+3,(0)}$ as in (1.15). The *secant linear virtual dimension* of $|D|$ is the number

$$\sigma\mathrm{ldim}(D) := \sum_{I,\sigma_t} (-1)^{|I|} \binom{n + k_{I,\sigma_t} - r_{I,\sigma_t} - 1}{n}, \tag{1.18}$$

where the sum ranges over all indexes $I \subset \{1, \ldots, n+3\}$ and t such that $0 \leq t \leq l+\epsilon$, $n = 2l + \epsilon$ and $0 \leq |I| \leq n - 2t$. The integers k_{I,σ_t} and r_{I,σ_t} are defined in (1.6) and in (1.5) respectively.

Conjecture 1.16. *The Euler characteristic of the divisor $\tilde{D}^\sigma$, defined in (1.17) as the strict transform of D in Y^σ, is*

$$\chi(Y^\sigma, \mathcal{O}_{Y^\sigma}(\tilde{D}^\sigma)) = \sigma\mathrm{ldim}(D).$$

The above questions are related to the *dimensionality problem* for linear systems of the form (1.1) and in particular to the Fröberg-Iarrobino conjectures [27, 33], which give a predicted value for the Hilbert series of an ideal generated by s general powers of linear forms in the polynomial ring with $n+1$ variables. We refer to [14, Section 2.1] for a more detailed account on this. In [14] the following conjectural answer to this problem was given in terms of Definition 1.15.

Conjecture 1.17 ([14, Conjecture 6.4])**.** *Let D be as in (1.15). Then*

$$h^0(X, \mathcal{O}_X(D)) = \max\{0, \sigma\mathrm{ldim}(D)\}.$$

Previous work [12, 24] contains a proof that Question 1.14 admits an affirmative answer, as well as proofs of Conjectures 1.16 and 1.17, for divisors satisfying the bound (1.14), namely those that do not contain positive multiples of the rational normal curve of degree n in the base locus, nor joins $\mathrm{J}(I, \sigma_t)$. The approach adopted was based on the study of the normal bundles of the exceptional divisors of linear cycles, L_I, and vanishing cohomologies of strict transforms on $X_{s,(n-2)}$, the blow-up along the linear cycles, see Theorem 1.10.

We believe that a proof of the above conjectures for the general case for $s = n+3$ would rely on the study of the normal bundles to the joins $\mathrm{J}(I, \sigma_t)$. We plan to develop this approach in future work.

On a more general note, we would like to point out that the construction of Y^σ and Question 1.14 and Conjectures 1.17 and 1.16 could be generalised to $\mathbb{P}^n$ blown-up in arbitrary number of points in linearly general position.

2. l-VERY AMPLE DIVISORS ON $X_{s,(0)}$

Definition 2.1 ([8])**.** Let X be a complex projective smooth variety. For an integer $l \geq 0$, a line bundle $\mathcal{O}_X(D)$ on X is said to be *l-very ample*, if for any 0-dimensional subscheme $Z \subset X$ of weight $h^0(Z, \mathcal{O}_Z) = l + 1$, the restriction map $H^0(X, \mathcal{O}_X(D)) \to H^0(Z, \mathcal{O}_X(D)|_Z)$ is surjective.

We will now recall some of the results obtained in the study of positivity of blown-up spaces or surfaces. Di Rocco [23] classified l-very ample line bundles on del Pezzo surfaces, namely for $\mathbb{P}^2$ blown-up at $s \leq 8$ points in general position. For general surfaces, very ample divisors on rational surfaces were considered by Harbourne [30]. De Volder and Laface [22] classified l-very ample divisors, for $l = 0, 1$, on the blow-up of $\mathbb{P}^3$ at s arbitrary general points. Ampleness and very ampleness properties of divisors on blow-ups at points of higher dimensional projective spaces in the case of points of multiplicity one were studied by Angelini [1], Ballico [3] and Coppens [21].

Positivity properties for blown-up $\mathbb{P}^n$ in general points were considered by Castravet and Laface. In particular, for small number of points in general position, $s \leq 2n$, the semi-ample and nef cones, that we describe in this paper in Corollary 3.2, were obtained via a different technique (private communication).

For any effective divisor on $X_{s,(0)}$ denote by $s(d)$ the number of points in $\mathcal{S}$ of which the multiplicity equals d. We introduce the following integer (see [12, Theorem 5.3]):

$$b = b(D) := \min\{n - s(d); s - n - 2\}. \tag{2.1}$$

We can describe l-very ample line bundles over $X_{s,(0)}$, the blown-up projective space at s points in general position, whose underlying divisor is of the form (1.2)

$$D = dH - \sum_{i=1}^{s} m_i E_i,$$

as follows.

Theorem 2.2 (l-very ample line bundles)**.** *Assume that $\mathcal{S} \subset \mathbb{P}^n$ is a collection of points in general position. Let l be a non-negative integer. Assume that either $s \leq 2n$ or $s \geq 2n + 1$ and d large enough, namely*

$$\sum_{i=1}^{s} m_i - nd \leq b - 1 - l, \tag{2.2}$$

where b is defined as in (2.1). Then a divisor D of the form (1.2) is l-very ample if and only if

$$\begin{aligned} &l \leq m_i, \ \forall i \in \{1, \dots, s\}, \\ &l \leq d - m_i - m_j, \ \forall i, j \in \{1, \dots, s\}, \ i \neq j. \end{aligned} \tag{2.3}$$

Remark 2.3. When $l = 0$ ($l = 1$), l-very ampleness corresponds to *global generation*, or *spannedness* (resp. very ampleness).

Remark 2.4. One can view conditions (1.3) as particular case of (2.3), by setting $l = -1$.

Corollary 2.5 (Globally generated line bundles)**.** *In the same notation of Theorem 2.2, assume that either $s \leq 2n$ or $s \geq 2n+1$ and*

$$\sum_{i=1}^{s} m_i - nd \leq b - 1.$$

Then D is globally generated if and only if

$$\begin{aligned} &0 \leq m_i, \ \forall i \in \{1, \dots, s\}, \\ &0 \leq d - m_i - m_j, \ \forall i, j \in \{1, \dots, s\}, \ i \neq j. \end{aligned} \tag{2.4}$$

Corollary 2.6 (Very ample line bundles)**.** *In the same notation of Theorem 2.2, assume that either $s \leq 2n$ or $s \geq 2n+1$ and*

$$\sum_{i=1}^{s} m_i - nd \leq b - 2.$$

Then D is very ample if and only if

$$\begin{aligned} &1 \leq m_i, \ \forall i \in \{1, \dots, s\}, \\ &1 \leq d - m_i - m_j, \ \forall i, j \in \{1, \dots, s\}, \ i \neq j. \end{aligned} \tag{2.5}$$

Example 2.7. We present an example where the bound (2.3) of Theorem 2.2 is sharp. Let us consider the anticanonical divisor of the blown-up $\mathbb{P}^2$ in eight points in general position

$$D := 3H - E_1 - \ldots - E_8.$$

Sections of D correspond to planar cubics passing through eight simple points. It is well-known that all such cubics meet at a ninth point, therefore D is not a globally generated divisor. However, D is nef.

2.1. **Proof of Theorem 2.2.** In order to give a proof of Theorem 2.2, we first give the following vanishing theorem, that has its own intrinsic interest.

Theorem 2.8. *In the same notation as Theorem 2.2, fix integers $d, m_1, \dots, m_s, l \geq 0$, $s \geq 1$. Assume that either $s \leq 2n$ or that $s \geq 2n+1$ and that (2.2) is satisfied. Moreover, assume that*

$$\begin{aligned} &l \leq m_i, \ \forall i \in \{1, \dots, s\}, \\ &l \leq d - m_i - m_j, \ \forall i, j \in \{1, \dots, s\}, \ i \neq j. \end{aligned} \tag{2.6}$$

Then $h^1(D \otimes \mathcal{I}_{\{q^{l+1}\}}) = 0$ for any $q \in X_{s,(0)}$.

Proof. Case (1). Assume first of all that $q \in E_i$, for some $i \in \{1, \dots, s\}$. We claim that

$$h^1(D \otimes \mathcal{I}_{\{q^{l+1}\}}) \leq h^1(D - (l+1)E_i). \tag{2.7}$$

Hence we conclude because the latter vanishes, by Theorem 1.3. We now prove that (2.7) holds. Let π be the blow-up of $X_{s,(0)}$ at $q \in E_i$ and let E_q be the exceptional divisor created. By the *projection formula* we have $H^i(D \otimes \mathcal{I}_{\{q^{l+1}\}}) \cong H^i(\pi^*(D) - (l+1)E_q)$. For $l = 0$, consider the exact sequence

$$0 \to \pi^*(D) - \pi^*(E_i) \to \pi^*(D) - E_q \to (\pi^*(D) - E_q)|_{\pi^*(E_i) - E_q} \to 0. \tag{2.8}$$

Notice that $\pi^*(E_i) - E_q$ is the blow-up of $E_i \cong \mathbb{P}^{n-1}$ at the point q: denote by h, e_q the generators of its Picard group. We have $(\pi^*(D) - E_q)|_{\pi^*(E_i) - E_q} \cong m_i h - e_q$,

in particular it has vanishing first cohomology group. Hence, looking at the long exact sequence in cohomologies associated with (2.8), one gets that the map

$$H^1(\pi^*(D) - \pi^*(E_i)) \to H^1(\pi^*(D) - E_q)$$

is surjective, therefore $h^1(\pi^*(D) - \pi^*(E_i)) \geq h^1(\pi^*(D) - E_q)$. Finally, by the projection formula one has $H^i(\pi^*(D) - \pi^*(E_i)) = H^i(D - E_i)$, so we conclude. For $l \geq 1$, one can iterate l times the above argument and conclude.

Case (2). Assume $q \in X_{s,(0)} \setminus \{E_1, \dots, E_s\}$. Hence q is the pull-back of a point $q' \in \mathbb{P}^n \setminus \{p_1, \dots, p_s\}$.

We will prove the statement by induction on n. The case $n = 1$ is obvious. Indeed, any such $D \otimes \mathcal{I}_{\{q^{l+1}\}}$ corresponds to a linear series on the projective line given by three points whose sum of the multiplicities is bounded above as follows $m_1 + m_2 + (l+1) \leq d + 1$. Hence the first cohomology group vanishes. From now on we will assume $n \geq 2$.

Recall that a set of points $\mathcal{S}$ of $\mathbb{P}^n$ is said to be in *linearly general position* if for each integer r we have $\sharp(S \cap L) \leq r + 1$, for all r-dimensional linear subspaces L in $\mathbb{P}^n$.

Case (2.a). Assume first that the points in $\mathcal{S} \cup \{q'\}$ are not in linearly general position in $\mathbb{P}^n$. If $s \geq n$, q' lies on a hyperplane H of $\mathbb{P}^n$ spanned by n points of $\mathcal{S}$. Reordering the points if necessary, assume that $q' \in H := \langle p_1, \dots, p_n \rangle$. If $s < n$, let H be any hyperplane containing $\mathcal{S} \cup \{q'\}$. Let $\bar{H}$ denote the pull-back of H on $X_{s,(0)}$. By Remark 1.8 part (1), $\bar{H}$ is isomorphic to the space $\mathbb{P}^{n-1}$ blown-up at $\bar{s} := \min\{s, n\}$ distinct points in general position, that we may denote by $\bar{H} \cong X^{n-1}_{\bar{s},(0)}$. Its Picard group is generated by $h := H|_H$, $e_i := E_i|_H$. As a divisor, we have $\bar{H} = H - \sum_{i=1}^{\bar{s}} E_i$. Consider the restriction exact sequence of line bundles

$$0 \to (D - \bar{H}) \otimes \mathcal{I}_{\{q^l\}} \to D \otimes \mathcal{I}_{\{q^{l+1}\}} \to (D \otimes \mathcal{I}_{\{q^{l+1}\}})|_{\bar{H}} \to 0. \tag{2.9}$$

We iterate this restriction procedure $l+1$ times. The restriction of the $(\lambda+1)$st exact sequence, $0 \leq \lambda \leq l$, is the complete linear series on $X^{n-1}_{\bar{s},(0)}$ given by

$$\left((d - \lambda)h - \sum_{i=1}^{\bar{s}} (m_i - \lambda) e_i \right) \otimes \mathcal{I}_{\{q^{l+1-\lambda}\}|_H}. \tag{2.10}$$

We leave it to the reader to verify that it satisfies the hypotheses of the theorem, for every $0 \leq \lambda \leq l$. Hence we conclude, by induction on n, that the first cohomology group vanishes.

Moreover, the (last) kernel is the line bundle associated with the following divisor:

$$d'H - \sum_{i=1}^{d} m'_i E_i := (d - l - 1)H - \sum_{i=1}^{\bar{s}} (m_i - l - 1)E_i - \sum_{i=\bar{s}+1}^{s} m_i E_i. \tag{2.11}$$

It is an easy computation to verify that the conditions of Theorem 1.3 are verified. Indeed, if $\bar{s} = s < n$ we have

$$\sum_{i=1}^{\bar{s}} m'_i - nd' = \sum_{i=1}^{\bar{s}} (m_i - l - 1) - n(d - l - 1) \leq 0,$$

because $m_i \le d$. Otherwise, if $\bar{s} = n \le s$, we compute

$$\sum_{i=1}^{s} m'_i - nd' = \sum_{i=1}^{\bar{s}} (m_i - l - 1) + \sum_{i=\bar{s}+1}^{s} m_i - n(d - l - 1) = \sum_{i=1}^{s} m_i - nd.$$

The above number is bounded above by 0 whenever $s \le 2n$, and by $b-1-l$ whenever $s \ge 2n+1$, by the hypotheses. Moreover, in all cases one has $m'_i + m'_j - d' \le 1$, for all $i \ne j$. Hence we conclude in this case, using Theorem 1.3.

Case (2.b). Lastly, assume that $\mathcal{S} \cup \{q'\}$ is in linearly general position in $\mathbb{P}^n$. If $s \ge n-1$, let H denote the hyperplane $\langle p_1, \ldots, p_{n-1}, q' \rangle$. If $s < n-1$, let H be any hyperplane containing $\mathcal{S} \cup \{q'\}$. In both cases such an H exists by the assumption that points of $\mathcal{S}$ are in general position. As in the previous case, let $\bar{H}$ denote the pull-back of H on $X_{s,(0)}$. It is isomorphic to the space $\mathbb{P}^{n-1}$ blown-up at $\bar{s} := \min\{s, n-1\}$ distinct points in general position, that we may denote by $\bar{H} \cong X^{n-1}_{\bar{s},(0)}$.

We iterate the same restriction procedure shown in (2.9) $l+1$ times as before. As before the restriction of the $(\lambda+1)$st exact sequence, that is of the form (2.10) with $\bar{s}$ differently defined here, verifies the hypotheses of the theorem, so it has vanishing first cohomology group by induction on n.

Furthermore, the (last) kernel, that is in the shape (2.11), with $\bar{s}, d', m'_i$ as defined here, verifies the conditions of Theorem 1.3. Indeed, if $\bar{s} = s < n-1$ then it is the same computation as before. While if $\bar{s} = n-1 \le s$ we have

$$\sum_{i=1}^{s} m'_i - nd' = \sum_{i=1}^{\bar{s}} (m_i - l - 1) + \sum_{i=\bar{s}+1}^{s} m_i - n(d - l - 1) = \sum_{i=1}^{s} m_i - nd + l + 1.$$

The number on the right hand side of the above expression is bounded above by 0 is $s \le 2n$ and by b if $s \ge 2n+1$. Hence we conclude. □

Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ be the linear system of the form (1.1).

Corollary 2.9. *Assume that $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ satisfies the conditions of Theorem 2.8. Then the linear system of elements of $\mathcal{L}$ that vanish with multiplicity $l+1$ at an arbitrary extra point, $\mathcal{L}_{n,d}(m_1, \ldots, m_s, l+1)$, is non-special.*

Proof. The projection formula implies that, for all $i \ge 0$, $H^i(X_{s,(0)}, D \otimes \mathcal{I}_{\{q^{l+1}\}}) \cong H^i(\mathbb{P}^n, \mathcal{L}_{n,d}(m_1, \ldots, m_s, l+1))$. Therefore $\mathcal{L}_{n,d}(m_1, \ldots, m_s, l+1)$ has the expected dimension. □

Before we proceed with the proof of the main result, Theorem 2.2, we need the following lemmas.

Lemma 2.10. *Let X be a complex projective smooth variety and $\mathcal{O}_X(D)$ a line bundle. Let Z be a 0-dimensional subscheme of X and let Z_0 be a flat degeneration of Z. Then $h^1(\mathcal{O}_X(D) \otimes \mathcal{I}_Z) \le h^1(\mathcal{O}_X(D) \otimes \mathcal{I}_{Z_0})$.*

Proof. It follows from the property of *upper semicontinuity* of cohomologies, see e.g, [31, Sect. III.12]. □

Lemma 2.11. *In the same notation of Lemma 2.10, let $Z_1 \subseteq Z_2$ be an inclusion of 0-dimensional schemes. Then $h^1(\mathcal{O}_X(D) \otimes \mathcal{I}_{Z_1}) \le h^1(\mathcal{O}_X(D) \otimes \mathcal{I}_{Z_2})$.*

Proof. If $Z_1 = Z_2$ then equality obviously holds. We will assume $Z_1 \subsetneq Z_2$. Consider the following short exact sequence

$$0 \to \mathcal{O}_X(D) \otimes \mathcal{I}_{Z_2} \to \mathcal{O}_X(D) \otimes \mathcal{I}_{Z_1} \to \mathcal{O}_X(D) \otimes \mathcal{I}_{Z_1}|_{Z_2 \setminus Z_1} \to 0.$$

Consider the associated long exact sequence in cohomology. Since $h^1(\mathcal{O}_X(D) \otimes \mathcal{I}_{Z_1}|_{Z_2 \setminus Z_1}) = 0$, the map $H^1(\mathcal{O}_X(D) \otimes \mathcal{I}_{Z_2}) \to H^1(\mathcal{O}_X(D) \otimes \mathcal{I}_{Z_1})$ is surjective and this concludes the proof. □

Lemma 2.10 and Lemma 2.11 will allow to reduce the proof of l-very ampleness for divisors D to the computation of vanishing theorems of the first cohomology group of the sheaf associated with D tensored by the ideal sheaf of a collection of fat points, whose multiplicities sum up to $l+1$.

Proof of Theorem 2.2. We first prove that (2.3) is sufficient condition for D to be l-very ample. For every 0-dimensional scheme $Z \subset X_{s,(0)}$ of weight $l+1$, consider the exact sequence of sheaves

$$0 \to \mathcal{O}_{X_{s,(0)}}(D) \otimes \mathcal{I}_Z \to \mathcal{O}_{X_{s,(0)}}(D) \to \mathcal{O}_{X_{s,(0)}}(D)|_Z \to 0. \tag{2.12}$$

We will prove that $h^1(X_{s,(0)}, \mathcal{O}_{X_{s,(0)}}(D) \otimes \mathcal{I}_Z) = 0$. This will imply the surjectivity of the map $H^0(X_{s,(0)}, \mathcal{O}_{X_{s,(0)}}(D)) \to H^0(Z, \mathcal{O}_{X_{s,(0)}}(D)|_Z)$, by taking the long exact sequence in cohomology associated with (2.12).

Let Z_0 denote a flat degeneration of Z with support at the union of points $q_1, \dots, q_s, q_{s+1} \in X_{s,(0)}$ with $q_i \in E_i$, for all $i = 1, \dots, s$, and $q_{s+1} \in X_{s,(0)} \setminus \{E_1, \dots, E_s\}$. Since every exceptional divisor E_i as well as $X_{s,(0)} \setminus \{E_1, \dots, E_s\}$ are homogeneous spaces, in order to prove that $h^1(D \otimes \mathcal{I}_Z) = 0$ for all $Z \subset X_{s,(0)}$ 0-dimensional schemes of weight $l+1$, by Lemma 2.10 it is enough to prove that the same statement holds for every such Z_0.

Set μ_i to be the weight of the irreducible component of Z_0 supported at q_i, for all $i = 1, \dots, s, s+1$. One has that $\mu_i \geq 0$, $\sum_{i=1}^{s+1} \mu_i = l+1$. In order to prove that $h^1(D \otimes \mathcal{I}_{Z_0}) = 0$, by Lemma 2.11, it suffices to prove the a priori stronger statement that $h^1(\mathcal{O}_X(D) \otimes \mathcal{I}_{\bar{Z}_0}) = 0$, for every $\bar{Z}_0$ collection of fat points $\{q_1^{\mu_1}, \dots, q_s^{\mu_s}, q_{s+1}^{\mu_{s+1}}\}$. Indeed, the following containment of schemes supported at q_i holds for all $i = 1, \dots, s, s+1$: $Z_i|_{q_i} \subseteq \{q_i^{\mu_i}\}$.

Assume first of all that $\bar{Z}_0$ has support in one point $q = q_i$, namely $\mu_i = l+1$ for some $i \in \{1, \dots, s, s+1\}$ and $m_j = 0$ for all $j \neq i$. We have $h^1(D \otimes \mathcal{I}_{\bar{Z}_0}) = 0$ by Theorem 2.8.

Assume now that $\bar{Z}_0$ is supported in several points $q_1, \dots, q_s, q_{s+1}$. We want to prove that $h^i(X_{s,(0)}, D \otimes \mathcal{I}_{\bar{Z}_0}) \leq h^i(X_{s,(0)}, (D - \sum_{i=1}^{s} \mu_i E_i) \otimes \mathcal{I}_{\{q_{s+1}^{\mu_{s+1}}\}}) = 0$. The first inequality follows from (2.7). The equality follows from Theorem 2.8; we leave it to the reader to verify that the conditions are indeed satisfied.

We now prove that (2.3) is necessary condition for D to be l-very ample, by induction on l.

Let us first assume $l = 0$, namely that D is base point free. If $m_i < 0$ then $m_i E_i$ would be contained in the base locus of D. If $m_i + m_j > d$ for some $i \neq j$, then the strict transform of the line $\langle p_i, p_j \rangle \subset \mathbb{P}^n$ would be contained in the base locus of D. In both cases we would obtain a contradiction.

Assume that $l = 1$, namely that D is very ample. If $m_i \leq 0$ (or $0 \leq d - m_i - m_j$ for some $i \neq j$), then E_i (resp. the strict transform of the line through p_i and p_j) would be contracted by D, a contradiction.

More generally, assume that D is l-very ample and $l \geq 2$. Then conditions (2.3) are satisfied. Indeed, if $m_i \leq l - 1$ for some i, we can find a 0-dimensional scheme, Z, of weight $l + 1$ such that $h^1(D \otimes \mathcal{I}_Z) > 0$. Let $Z \subset E_i$ be a *l-jet scheme* centred at $q \in E_i$ (see [43]). Consider the restriction $D \otimes \mathcal{I}_Z|_{E_i} \cong m_1 h \otimes \mathcal{I}_Z$, where h is the hyperplane class of $E_i \cong \mathbb{P}^{n-1}$. We have $h^1(E_i, D \otimes \mathcal{I}_Z|_{E_i}) \geq 1$, hence $h^1(X_{s,(0)}, D \otimes \mathcal{I}_Z) \geq 1$. To see this, let $x_1 \ldots, x_{n-1}$ be affine coordinates for an affine chart $U \subset E_i$ and let Z be the jet-scheme with support $q = (0, \ldots, 0) \in U$ given by the tangent directions up to order l along x_1. The space of global sections of $D \otimes \mathcal{I}_Z|_{E_i}$ is isomorphic to the set of degree-m_i polynomials $f(x_1, \ldots, x_{n-1})$, whose partial derivatives $\partial^\lambda f / \partial x_1^\lambda$ vanish at q, for $0 \leq \lambda \leq l$. On the other hand, $H^1(E_i, D \otimes \mathcal{I}_Z|_{E_i})$ is the space of linear dependencies among the $l + 1$ conditions imposed by the vanishing of the partial derivatives to the coefficients of f. Since $m_i \leq l - 1$ then f is a polynomial of degree bounded by $l - 1$, therefore $\partial^l f / \partial x_1^l \equiv 0$ for every such a polynomial, and we conclude.

Similarly, if $d - m_i - m_j \leq l - 1$ for some $i, j, i \neq j$, then one finds a jet-scheme Z contained in the pull-back of the line through p_i and p_j, L, for which $h^1(X_{s,(0)}, D \otimes \mathcal{I}_Z) \geq 1$. Indeed, if Z is such a scheme, then the restriction is $D \otimes \mathcal{I}_Z|_L \cong (d - m_i - m_j) h \otimes \mathcal{I}_Z|_L$, where in this case h is the class of a point on L, and $Z|_L$ is a fat point of multiplicity l on L. One concludes by the Riemann-Roch Theorem that $h^1(L, D \otimes \mathcal{I}_Z|_L) \geq 1$ because $\chi(L, D \otimes \mathcal{I}_Z|_L) = (d - m_i - m_j) - l \leq -1$ and $h^0(L, D \otimes \mathcal{I}_Z|_L) = 0$.

□

2.2. **l-jet ampleness.** In [8], Beltrametti, Francia and Sommese introduced notions of higher order embeddings, one of these being l-very ampleness (Definition 2.1) with the aim of studying the *adjoint bundle* on surfaces.

Definition 2.12. In the same notation as Definition 2.1, if for any fat point $Z = \{q^{l+1}\}$, $q \in X$, the natural restriction map to Z, $H^0(X, \mathcal{O}_X(D)) \to H^0(Z, \mathcal{O}_X(D)|_Z)$, is surjective, then D is said to be *l-jet spanned*.

Moreover, if for any collection of *fat points* $Z = \{q_1^{\mu_1}, \ldots, q_\sigma^{\mu_\sigma}\}$ such that $\sum_{i=1}^{\sigma} \mu_i = l + 1$, the restriction map to Z is surjective, then D is said to be *l-jet ample.*

Remark 2.13. Theorem 2.8 can be restated in terms of l-jet spannedness. Namely any divisor D satisfying the hypotheses is l-jet spanned.

Proposition 2.14 ([9, Proposition 2.2])**.** *In the above notation, if D is l-jet ample, then D is l-very ample.*

The converse of Proposition 2.14 is true for the projective space $\mathbb{P}^n$ and for curves, but not in general. In this section we proved that the converse is true for lines bundle D on $X_{s,(0)}$, that satisfy the hypotheses of Theorem 2.2.

Theorem 2.15. *Assume that $s \leq 2n$, or $s \geq 2n + 1$ and* (2.2)*. Assume that D is a line bundle on $X_{s,(0)}$ of the form* (1.2)*. The following are equivalent:*

(1) D satisfies (2.3)*;*
(2) D is l-jet ample;
(3) D is l-very ample.

Proof. We proved that the natural restriction map of the global sections of D to the any fat point of multiplicity $l+1$ is surjective in Theorem 2.8, see also Remark 2.13. We showed that the same is true in the case of arbitrary collections of fat points whose multiplicity sum up to $l+1$ in the first part of the proof of Theorem 2.2. This proves that (1) implies (2). Moreover, (2) implies (3) by Proposition 2.14. Finally, that (3) implies (1) was proved in the second part of the proof of Theorem 2.2. □

3. Other positivity properties of divisors on $X_{s,(0)}$

In this section we will apply Theorem 2.2 to establish further positivity properties of divisors on $X_{s,(0)}$. All results we prove in this section apply to $\mathbb{Q}-$divisors on the blown-up projective space.

We recall here the notation introduced in Section 2: for a given divisor D of the form $D = dH - \sum_{i=1}^{s} m_i E_i$, (cfr. (1.2)), we will use the integer $b = b(D) := \min\{n - s(d); s - n - 2\}$, defined in (2.1) and the bound (2.2):

$$\sum_{i=1}^{s} m_i - nd \leq b - 1 - l.$$

3.1. **Semi-ampleness and ampleness.** A line bundle is ample if some positive power is very ample. It is known that for smooth toric varieties a divisor is ample if and only if is very ample and nef if and only if is globally generated. From Corollary 2.5 and Corollary 2.6, we obtain that this holds for a small number of points $s \leq 2n$ too, as well as for arbitrary s under a bound on the coefficients.

A line bundle is called *semi-ample*, or *eventually free*, if some positive power is globally generated. By (2.4), one can see that a divisor is semi-ample if and only if it is globally generated.

Theorem 3.1. *Let $X_{s,(0)}$ be defined as in Section 1. Assume $s \leq 2n$.*

(1) The cone of semi-ample divisors in $\mathrm{N}^1(X_{s,(0)})_{\mathbb{R}}$ is given by (2.4).
(2) The cone of ample divisors in $\mathrm{N}^1(X_{s,(0)})_{\mathbb{R}}$ is given by (2.5).

Assume $s \geq 2n+1$.

(1) Divisors satisfying (2.2) with $l = 0$ are semi-ample if and only if (2.4).
(2) Divisors satisfying (2.2) with $l = 1$ are ample if and only if (2.5).

3.2. **Nefness.** For any projective variety, Kleiman [37] showed that a divisor is ample if and only if its numerical equivalence class lies in the interior of the nef cone (see also [39, Theorem 1.4.23]).

For a line bundle, being generated by the global sections implies being nef, but the opposite is not true in general, see e.g. Example 2.7. However for line bundles on $X_{s,(0)}$, with $s \leq 2n$, or with arbitrary s under a bound on the coefficients, these two properties are equivalent.

Theorem 3.2. *In the same notation as Theorem 2.2, assume that for D of the form (1.2) we have that either $s \leq 2n$ or $s \geq 2n+1$ and (2.2) with $l = 0$ is satisfied. Then D is nef if and only if is globally generated.*

In particular if $s \leq 2n$, the nef cone of $X_{s,(0)}$ is given by (2.4).

Proof. If D is nef, then for effective 1-cylce C, $D \cdot C \geq 0$. In particular the divisor D intersects positively the classes of lines through two points and classes of lines

in the exceptional divisors. This means inequalities (2.4) hold and therefore the divisor D is globally generated by Corollary 2.5. □

Corollary 3.3. *The nef cone and the cone of semi-ample divisors on $X_{s,(0)}$, for $s \leq 2n$, coincide.*

Corollary 3.4. *The Mori Cone of curves of the blown-up $\mathbb{P}^n$ in $s \leq 2n$ points, $\mathrm{NE}(X_{s,(0)})$, is generated by the classes of lines through two points and the classes of lines in the exceptional divisors.*

3.3. **Bigness.** The *pseudo-effective cone* is the closure of the effective cone and the *big cone* is the interior of the pseudo-effective cone (see e.g. [39, Theorem 2.2.26]). For Mori Dream Spaces, the effective cone is polyhedral and closed, hence it coincides with the pseudo-effective cone. In particular, when equations for the supporting hyperplanes of such a cone are known, one can recover the big cone.

We consider here $\mathbb{P}^n$ blown-up in $n+3$ general points, that is known to be a Mori Dream Space (see e.g. [17]). For $|I| \leq n-2t$ the number k_{I,σ_t} (1.6), introduced in Section 1.2, is the multiplicity of containment of the join $\mathrm{J}(I,\sigma_t)$ in the base locus of D, see Lemma 1.5. Theorem 1.7 states that the equation $k_{I,\sigma_t}=0$, for $|I|=n-2t+1$, gives a supporting hyperplane for the pseudo-effective cone of $X_{n+3,(0)}$, while $k_{I,\sigma_t}=0$, for $|I|=n-2t$, gives a supporting hyperplane for the movable cone of $X_{n+3,(0)}$. The next results shows that $k_{I,\sigma_t}=-1$, for $|I|=n-2t+1$, is a supporting hyperplane for the big cone of $X_{n+3,(0)}$.

Theorem 3.5. *Assume $s \leq n+2$. The big cone of $X_{s,(0)}$ is described by*

$$\begin{aligned} & d>0, \\ & m_i<d,\ \forall i \in \{1,\ldots,s\}, \\ & \sum_{i\in I} m_i < nd,\ \forall I \subseteq \{1,\ldots,s\},\ |I|=n+1,n+2. \end{aligned} \tag{3.1}$$

Assume $s=n+3$. The big cone of $X_{s,(0)}$ is described by

$$\begin{aligned} & d>0, \\ & m_i<d,\ \forall i \in \{1,\ldots,s\}, \\ & \sum_{i\in I} m_i < nd,\ \forall I \subseteq \{1,\ldots,s\},\ |I|=n+1,n+2, \\ & k_{I,\sigma_t}<0,\ \forall |I|=n-2t+1,\ 1\leq t\leq \nu+\epsilon, \end{aligned} \tag{3.2}$$

where $n=2\nu+\epsilon$ with $\epsilon\in\{0,1\}$.

3.4. **Fujita's conjectures hold the blown-up $\mathbb{P}^n$ in points.**

Conjecture 3.6 (Fujita's conjectures, [25])**.** *Let X be an n-dimensional projective algebraic variety, smooth or with mild singularities. Let K_X be the canonical divisor of X and D an ample divisor on X. Then the following holds.*

(1) For $m \geq n+1$, $mD+K_X$ is globally generated.
(2) For $m \geq n+2$, $mD+K_X$ is very ample.

In [44] Payne established the Fujita's conjecture for toric varieties, in particular for $X_{s,(0)}$, for $s \leq n+1$. Using the results from this article, we can extend the above to $s \leq 2n$.

Proposition 3.7. *Let $X_{s,(0)}$ be the blown-up $\mathbb{P}^n$ at s points in general position with $s \leq 2n$. Conjecture 3.6 holds for $X_{s,(0)}$.*

Proof. For smooth varieties, a consequence of the Mori's Cone Theorem is that the same statement of Conjecture 3.6 with globally generated (very ample) replaced by nef (resp. ample) holds (see also [25]).

For $X_{s,(0)}$, $s \leq 2n$, global generation (very ampleness) is equivalent to nefness (resp. ampleness), by Theorem 3.1 and Theorem 3.2. This concludes the proof. □

Proposition 3.8. *Let $X_{s,(0)}$ be the blown-up $\mathbb{P}^n$ in an arbitrary number of points in general position, s, and let D be a divisor on $X_{s,(0)}$ such that*

$$\sum_{i=1}^{s} m_i \leq nd \tag{3.3}$$

Then Conjecture 3.6 holds for D.

Proof. It is enough to consider the case $s \geq 2n+1$. Write $X = X_{s,(0)}$. Notice that the divisor $mD + K_X$ has the following properties

$$\begin{aligned}
\sum_{i=1}^{s}(mm_i - n + 1) - n(md - n - 1) &= m(\sum_{i=1}^{s} m_i - nd) + n(n+1) - s(n-1) \\
&\leq n(n+1) - (2n+1)(n-1) \\
&= (-n^2 + n) + (n+1) \\
&\leq -2 + n + 1 \\
&= n - 1.
\end{aligned}$$

Notice that $b(mD + K_X) = n - 1$, for $s = 2n+1$ and $b(D) = n$ for $s \geq 2n+2$, using the definition (2.1). Therefore $mD + K_X$ satisfies conditions of Theorem 3.1. We now leave it to the reader to check that if D is ample then the divisor $mD + K_X$ satisfies conditions (2.4) and (2.5).

□

4. Globally generated divisors on $X_{s,(r)}$

Let D be a divisor on $X_{s,(0)}$ of the form (1.2) and let $D_{(r)}$ be its strict transform of in $X_{s,(r)}$, as in (1.13):

$$D_{(r)} = dH - \sum_{\substack{I \subset \{1,\dots,s\}: \\ 0 \leq |I| \leq r+1}} k_I E_I.$$

If $\bar{r}$ is the dimension of the linear base locus of D, denote $\tilde{D} := D_{(\bar{r})}$ (see also Section 1).

Theorem 4.1. *Assume that $\mathcal{S} \subset \mathbb{P}^n$ is a collection of points in general position. Assume that $s \leq n+1$ or that $s \geq n+2$ and d is large enough, namely*

$$\sum_{i=1}^{s} m_i - nd \leq \max\{0, s - n - 3\}. \tag{4.1}$$

Then for any $0 \leq r \leq n-1$ the divisor $D_{(r)}$ on $X_{s,(r)}$ is globally generated if and only if

$$\begin{aligned} &0 \leq m_i \leq d, \ \forall i \in \{1, \ldots, s\}, \\ &0 \leq (r+1)d - \sum_{i \in I} m_i, \ \forall I \subseteq \{1, \ldots, s\}, \ |I| = r+2. \end{aligned} \tag{4.2}$$

Remark 4.2. Notice that if $s \leq \left\lfloor \frac{r+2}{r+1} n \right\rfloor$, then the condition on the degree (4.1) is always satisfied. Hence in this range, Theorem 4.1 provides a complete classification of divisors D on $X_{s,(0)}$ whose strict transform $D_{(r)}$ in $X_{s,(0)}$ is globally generated.

Remark 4.3. If $\bar{r} = n-1$, then $\tilde{D}$ is the strict transform of $D_{(n-2)}$ via the contraction of strict transforms of hyperplanes $L_{I(n)} \subset \mathbb{P}^n$, $X_{s,(n-2)} \dashrightarrow X_{s,(n-2)}$, cfr. Remark 1.9. Therefore in the same assumptions of Theorem 4.1, we have that $\tilde{D}$ is globally generated if and only if

$$\begin{aligned} &0 \leq m_i \leq d, \ \forall i \in \{1, \ldots, s\}, \\ &0 \leq nd - \sum_{i \in I} m_i, \ \forall I \subseteq \{1, \ldots, s\}, \ |I| = n+1. \end{aligned} \tag{4.3}$$

In order to prove the main result of this section, Theorem 4.1, we need the following result.

Lemma 4.4. *In the above notation, assume*

$$\sum_{i=1}^{s} m_i - nd \leq \max\{0, s-n-3\}. \tag{4.4}$$

Then for any point $q \in E_I$, for some E_I, we have

$$h^1(D_{(r)} \otimes \mathcal{I}_q) \leq h^1(D_{(r)} - E_I).$$

Proof. We prove the statement by induction on n. If $n = 2$, it follows from Theorem 2.8 applied in the case with $l = 0$. We will assume $n \geq 3$. Consider the short exact sequence

$$0 \to D_{(r)} - E_I \to D_{(r)} \otimes \mathcal{I}_q \to D_{(r)} \otimes \mathcal{I}_q|_{E_I} \to 0. \tag{4.5}$$

We claim that

$$h^1(D_{(r)} \otimes \mathcal{I}_q|_{E_I}) = 0. \tag{4.6}$$

The statement will follow from this, by looking at the long exact sequence in cohomology associated with (4.5).

We now prove the claim, namely that (4.6) holds. Set $\rho+1 := \#I$. We recall that E_I is a product, whose second factor is isomorphic to $X^{n-\rho-1}_{\alpha(I),(r-\rho-1)}$, the blown-up projective space of dimension $n-\rho-1$ along linear cycles up to dimension $r-\rho-1$ spanned by $\alpha(I)$ points. We refer to [24, Lemma 2.5] for details. We introduce the positive integer

$$\alpha(I) := \#\{I(\rho+1) : k_{I(\rho+1)} \geq 1, \ I(\rho+1) \supset I\},$$

where $I(\rho+1)$ denotes an index set contained in $\{1, \ldots, s\}$ of cardinality $\rho+2$ and $k_{I(\rho+1)}$ is defined as in (1.4). Moreover, let F be a divisor on the blow-up of $\mathbb{P}^{n-\rho-1}$

at $\alpha(I)$ points in linearly general position, $X^{n-\rho-1}_{\alpha(I),(0)}$, of the following form:

$$F := k_I h - \sum_{\substack{I(\rho+1)\supset I:\\ k_{I(\rho+1)}\geq 1}} k_{I(\rho+1)} \cdot e_{I(\rho+1)|I}. \tag{4.7}$$

We claim the following hold.

(1) The restriction of $D_{(r)}$ to E_I is $D_{(r)}|_{E_I} = (0, F_{(r-\rho-1)})$, where $F_{(r-\rho-1)}$ denotes the strict transform of F in $X^{n-\rho-1}_{\alpha(I),(r-\rho-1)}$.

(2) If D satisfies the bound (4.4), so does F.

The proof of (4.6) follows from these two claims. Indeed (1) implies that

$$\mathcal{O}_{X_{(r)}}(D_{(r)}) \otimes \mathcal{I}_q|_{E_I} \cong \mathcal{O}_{X^{n-\rho-1}_{\alpha(I),(r-\rho-1)}}(F_{(r-\rho-1)}) \otimes \mathcal{I}_{q_b},$$

where $q = (q_b, q_f) \in E_I$. This has vanishing first cohomology group, by induction on n, since F satisfies (4.4), by (2).

We are left to proof the two statements (1) and (2).

We first prove (1). The fact that the first factor of the restriction $D_{(r)}|_{E_I}$ is zero follows from the computation of normal bundles of the E_I's and from intersection theory on $X_{s,(r)}$, see [12, Section 4] or [24, Section 2]. We now compute the second factor. We have

$$E_I|_{E_I} = (*, h)$$
$$E_I|_{E_{I(j)}} = (*, e_{J|I}), \text{ for all } J \supset I,$$

where $*$ denotes the appropriate divisor on the first factor, as we are only interested here in the second factor. The classes of the divisors h and $e_{J|I}$ generate the Picard group of $X^{n-\rho-1}_{\alpha(I),(r-\rho-1)}$. Therefore one can compute

$$D_{(r)}|_{E_I} = \left(0, k_I h - \sum_{J\supset I} k_J e_{J|I}\right).$$

To conclude, we need to prove that the second factor on the right hand side of the above expression equals the strict transform $F_{(r-i-1)}$. Notice first that for any integer $\tau \geq 1$, one has $|I(\tau+\rho+1) \setminus I| = \tau + 1$. Hence it is enough to prove that

$$\sum_{\substack{I(\rho+1)\supset I:\\ I(\rho+1)\subset I(\tau+\rho+1)}} k_{I(\rho+1)} - \tau k_I = k_{I(\tau+\rho+1)}.$$

By definition, the left hand side equals

$$\begin{aligned}
&\sum_{I\subset I(\rho+1)\subset I(\tau+\rho+1)} k_I + \sum_{j\in I(\tau+\rho+1)\setminus I} m_j - (\tau+1)d - \tau k_I = \\
&= (1+\tau)k_I + \sum_{j\in I(\tau+\rho+1)\setminus I} m_j - (\tau+1)d - \tau k_I \\
&= k_I + \sum_{j\in I(\tau+\rho+1)\setminus I} m_j - (\tau+1)d \\
&= k_{I(\tau+\rho+1)}.
\end{aligned}$$

We now prove (2). We need to show that the following inequality holds

$$\sum_{I(\rho+1)\supset I} k_{I(\rho+1)} \leq (n-\rho-1)k_I + \max\{0, \alpha(I) - (n-\rho-1) - 3\}. \tag{4.8}$$

Set $A(I) = \{j \in S \setminus I : k_{I\cup\{j\}} \geq 1\}$. Notice that $\#A(I) = \alpha(I) \leq s-\rho-1$. Since the left hand side of (4.8) equals

$$\sum_{j\in A(I)} (k_I + m_j - d) = \alpha(I)k_I + \sum_{j\in A(I)} m_j - \alpha(I)d,$$

we need to prove the equivalent inequality

$$\sum_{j\in A(I)} m_j - \alpha(I)d \leq (n-\rho-1-\alpha(I))k_I + \max\{0, \alpha(I) - (n-\rho-1) - 3\}. \tag{4.9}$$

For $\alpha(I) \leq n-\rho-1$, this holds since $|A(I)| = \alpha(I)$ and $m_j \leq d$ so that the left hand side of (4.9) is a non-positive integer, while the right hand side is non-negative because $k_I \geq 0$. For $\alpha(I) \geq n-\rho$, the left hand side of (4.9) equals

$$\sum_{j\in S} m_j - nd - k_I - \sum_{j\in S\setminus(I\cup A(I))} m_j - (\alpha(I) - n + \rho)d,$$

therefore (4.9) is equivalent to

$$\begin{aligned}&(\alpha(I) - n + \rho)(k_I - d) + \left(\sum_{j\in S} m_j - nd\right) - \sum_{j\in S\setminus(I\cup A(I))} m_j\\ &\leq \max\{0, \alpha(I) - (n-\rho-1) - 3\}.\end{aligned}$$

One concludes by noticing that $k_I \leq d$ and by using (4.1); we leave the details to the reader.

□

Proof of Theorem 4.1. The case $r = 0$ is contained in Corollary 2.5. We will assume $r \geq 1$. We assume first that the inequalities (4.1) are satisfied and we claim $h^1(D_{(r)} \otimes \mathcal{I}_q) = 0$, for all points q. Hence $D_{(r)}$ is globally generated.

In order to show this, we distinguish the following two cases:

(1) q is on some exceptional divisor E_I,
(2) q is the pull-back of a point outside of the union $\sum_{I\subseteq\{1,\ldots,s\},|I|\leq r+1} L_I$.

Case (1). Assume $q \in E_I$ and write $\rho + 1 := |I|$, $\rho \leq r$. We want to prove that $h^1(D_{(r)} - E_I) = 0$. Then the conclusion will follow from Theorem 4.4.

Reordering the points if necessary, we may assume that $1 \in I$. Define the divisor $D' = D - E_1$ in $X_{s,(0)}$. One can easily check that D' satisfies the hypotheses of Theorem 1.10, therefore $h^1(D'_{(r)}) = 0$.

Consider the set $\mathcal{J}$ of all indices J of cardinality $1 \leq |J| \leq r+1$ such that $1 \in J$ and such that $\sum_{i\in J} m_i - |J|d \geq 0$. Let us endow $\mathcal{J}$ with the *graded lexicographical order* on the index sets, namely if $|J_1| < |J_2|$ then $J_1 \prec J_2$, while if $|J_1| = |J_2|$ we use the lexicographical order.

Notice that $I \in \mathcal{J}$ and that $D'_{(r)} = D_{(r)} - \sum_{J\in\mathcal{J}} E_J$. In other words,

$$D_{(r)} - E_I = D'_{(r)} + \sum_{\substack{J\in\mathcal{J}:\\ J\neq I}} E_J.$$

We can obtain $D'_{(r)}$ as the residual of iterative applications of exact sequences starting from $D'_{(r)}+\sum_{J\in\mathcal{J},J\neq I}E_J$ by restrictions to the exceptional divisors E_J, with $J\in\mathcal{J}$, $J\neq I$, following the order on $\mathcal{J}$. More precisely, one starts from $D'_{(r)}+\sum_{J\in\mathcal{J},J\neq I}E_J$ and restricts to E_1, then to all E_J's, with $J\in\mathcal{J}\setminus\{I\}$, $|J|=2$, then to all E_J's, with $J\in\mathcal{J}\setminus\{I\}$, $|J|=3$, etc. We claim that each restriction has vanishing first cohomology group. This gives a proof of the statement, since the last kernel, that is $D'_{(r)}$, has vanishing first cohomology too, by Theorem 1.10.

To prove the claim, recall that each exceptional divisor E_J is the product of two blown-up projective spaces of dimension $|J|-1$ and $n-|J|$ respectively, see [24, Lemma 2.5]. In particular the first component is isomorphic to $\mathbb{P}^{|J|-1}$ blown-up along linear cycles (see Remark 1.8). Let us denote by h the class of a general hyperplane on the first component and by $e_{J'}$ the class of the restriction of the exceptional divisors $E_{J'}$, namely $E_{J'}|_{E_J}=(e_{J'},0)$, for all $J'\subset J$. Let $\mathrm{Cr}(h)$ be the proper transform of the *standard Cremona transformation* of the hyperplane class h on $\mathbb{P}^{|J|-1}$, i.e.

$$\mathrm{Cr}(h)=(|J|-1)h-\sum_{\substack{J'\subset J:\\ |J'|<|J|-1}}(|J|-|J'|-1)e_{J'}. \tag{4.10}$$

We have

$$\begin{aligned} D'_{(r)}|_{E_J}&=(0,*)\text{ for every }E_J,\\ E_J|_{E_J}&=(-\mathrm{Cr}(h),*),\\ E_{J'}|_{E_J}&=(0,*)\text{ for all }J'\supset J,\end{aligned}$$

where we use $*$ to denote the appropriate divisor on the second factor. See [24, Sect. 2-3] for details.

Therefore each of the above restrictions is

$$\left(D'_{(r)}+\sum_{\substack{J'\in\mathcal{J}:\\ J'\neq I,J\prec J'}}E_{J'}\right)|_{E_J}=(-\mathrm{Cr}(h),*).$$

It has vanishing first cohomology group by [24, Theorem 3.1]. This concludes the proof of Case (1).

Case (2). In this case q is the pull-back of a point $q'\in\mathbb{P}^n\setminus\bigcup_{I\subset\{1,\dots,s\},|I|\leq r}L_I$. As in the proof of Theorem 2.8 we distinguish two subcases and we prove the claim by induction on n. The case $n=1$ is obvious. Assume $n\geq 2$.

Case (2.a). Let us assume first that the points $p_1,\dots,p_s,q'$ are not in linearly general position. If $s\geq n$, q' lies on a hyperplane H of $\mathbb{P}^n$ spanned by n points of $\mathcal{S}$. Reordering the points if necessary, assume that $q'\in H:=\langle p_1,\dots,p_n\rangle$. If $s<n$, let H be any hyperplane containing $\mathcal{S}\cup\{q'\}$. Let $\bar{H}$ denote the pull-back of H on $X_{s,(r)}$. It is isomorphic to the space $\mathbb{P}^{n-1}$ blow-up along linear cycles of dimension up to $\min\{r,n-2\}$, spanned by $\bar{s}:=\min\{s,n\}$ points in general position, that we

may denote by $\bar{H} \cong X^{n-1}_{\bar{s},(r)}$ as in Remark 1.8, part (1). As a divisor, we have

$$\bar{H} = H - \sum_{i=1}^{\bar{s}-1} E_i - \sum_{\substack{I\subset\{1,\ldots,\bar{s}\}:,\\ 1\le|I|\le\min\{r,n-2\}}} E_I. \tag{4.11}$$

Consider the restriction exact sequence of line bundles

$$0 \to D_{(r)} - \bar{H} \to D_{(r)} \otimes \mathcal{I}_q \to (D_{(r)} \otimes \mathcal{I}_q)|_{\bar{H}} \to 0. \tag{4.12}$$

The restriction, $D_{(r)}|_{\bar{H}} \otimes \mathcal{I}_q$, is a *toric divisor* on the blown-up space $\bar{H} \cong X^{n-1}_{\bar{s},(r)}$, with a point, q, in possible special linear configuration with the other points. As in the proof of Theorem 2.8, Case (2.a), we conclude that it has vanishing first cohomology group by induction on n. The kernel also has vanishing first cohomology because it has possibly only simple linear obstructions, see [24, Theorem 1.5]. We conclude that $h^1(D_{(r)} \otimes \mathcal{I}_q) = 0$.

Case (2.b). Let us assume now that the points $p_1, \ldots, p_s, q'$ are in linearly general position. If $s \ge n-1$, let H denote the hyperplane $\langle p_1, \ldots, p_{n-1}, q'\rangle$. If $s < n-1$, let H be any hyperplane containing $\mathcal{S} \cup \{q'\}$. In both cases such an H exists by the assumption that $\mathcal{S}$ is a set of points in general position. Let $\bar{H}$ denote the pull-back of H on $X_{s,(r)}$, as in (4.11) and consider the corresponding restriction sequence as in (4.12).

As in Case (2.a), we conclude by induction on n and by noticing that the kernel has only possibly simple linear obstructions.

Assume now that one of the inequalities in (4.2) does not hold. We claim that $D_{(r)}$ is not globally generated. Indeed, if $m_i \ge d+1$ then the divisor D is not effective therefore $D_{(r)}$ is not globally generated. If $m_i \le -1$ then the divisor E_i is in the base locus of $D_{(r)}$. If $k_I \ge 1$ for some I such that $|I| = r+2$ and $r \le n-1$, then the divisor $D_{(r)}$ contains in its base locus the strict transform of the linear cycle L_I by Lemma 1.4, therefore is not globally generated.

□

Remark 4.5. The strict transform $\tilde{D} = D_{(\bar{r})}$ of $\mathcal{L}$ is base point free if (4.2) is satisfied with $r = \bar{r}$.

4.1. **Vanishing Cohomology of strict transforms.** In this section we will determine the base locus and their intersection multiplicity for divisors satisfying condition (4.4). Furthermore, Theorem 4.9 answers Question 1.1 posed in [12] in this range.

Recall that a line bundle is globally generated if and only if the associated linear system is base point free. We are now ready to prove that Theorem 4.1 implies a complete description of the base locus of all non-empty linear systems in $\mathbb{P}^n$ of the form $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ (1.1) that are only linearly obstructed.

Theorem 4.6 (Base locus of linear systems)**.** *The base locus of the divisor $D_{(r)}$ on $X_{s,(r)}$, strict transform of $D = dH - \sum_{i=1}^{s} m_i E_i$ with (4.1), namely*

$$\sum_{i=1}^{s} m_i \le nd + \max\{0, s-n-3\},$$

is the formal sum

$$\sum_{\substack{I\subseteq\{1,\ldots,s\}:\\ |I|\geq r+1}} k_I L_I \in A^*(X_{s,(r)}). \tag{4.13}$$

In particular, if $s \leq n+2$, the sum (4.13) with $r=-1$, describes the base locus of all non-empty linear systems $\mathcal{L}_{n,d}(m_1,\ldots,m_s)$, while if $s\geq n+3$ the sum (4.13) with $r=-1$, describes the base locus of non-empty linear systems satisfying the bound (4.1).

Proof. By Lemma 1.4, each linear subspace L_I with $k_I>0$ is a base locus cycle for $\mathcal{L}$ and k_I is its exact multiplicity of containment. Therefore

$$\bigcup_{\substack{I\subseteq\{1,\ldots,s\}:\\ |I|\geq r+1}} k_I L_I \subset \mathrm{Bs}(|D_{(r)}|).$$

By Theorem 4.1, the strict transform $D_{(r)}$ of an element of $\mathcal{L}$ is base point free as soon as no higher dimensional cycle, i.e. no $(r+1)$-plane, is contained in the base locus, namely when $k_I\leq 0$ for all $I\subseteq\{1,\ldots,s\}$ of cardinality $r+1$. In particular, if $\bar{r}$ is the dimension of the linear base locus of D, then $\tilde{D}=D_{(\bar{r})}$ is base point free.

Since the total transform of $D_{(r)}$ in $X_{s,(\bar{r})}$ equals

$$\tilde{D}+\sum_{\substack{I\subseteq\{1,\ldots,s\}:\\ |I|\geq r+1}} k_I E_I,$$

one concludes that the base locus of D is supported only along linear cycles

$$\mathrm{Bs}(|D_{(r)}|)\subset \bigcup_{\substack{I\subseteq\{1,\ldots,s\}:\\ |I|\geq r+1}} k_I L_I.$$

□

Remark 4.7. The unique rational normal curve C of degree n through $n+3$ points of $\mathbb{P}^n$ and its secant varieties were studied in [14] as cycles of the base locus of non-empty linear systems $\mathcal{L}$, see also Section 3.3. Condition (4.1) for $s=n+3$ says that neither C nor $\sigma_t(C)$ are contained in the base locus of $\mathcal{L}$. Hence Theorem 4.6 states that if C is not in the base locus of $\mathcal{L}$, then nothing else is, besides the linear cycles.

Remark 4.8. Recall that for a line bundle $D_{(r)}$ as above, the *stable base locus* is defined as

$$\mathbb{B}(D_{(r)})=\cap_{m\in\mathbb{N}}\mathrm{Bs}(|mD_{(r)}|).$$

The obvious equality $k_I(mD)=mk_I(D)$, for any integer $m\geq 1$, and Theorem 4.6 show that $\mathrm{Bs}(|mD_{(r)}|)=m\cdot\mathrm{Bs}(|D_{(r)}|)$. Therefore the base locus of $D_{(r)}$ is stable, namely

$$\mathbb{B}(D_{(r)})=\mathrm{Bs}(|D_{(r)}|).$$

Because the stable base locus is invariant under taking multiples, we can extend this definition to the case of $\mathbb{R}$-divisors, see [11, Lemma 3.5.3]. In particular we can consider the stable base locus of $\epsilon D_{(r)}$, for any $\epsilon\in\mathbb{R}$, and obtain

$$\mathbb{B}(\epsilon D_{(r)})=\mathbb{B}(D_{(r)})=\mathrm{Bs}(|D_{(r)}|).$$

We now can prove that the strict transform of D in the iterated blow-up along its base locus has vanishing cohomology groups. This answers affirmatively Question 1.11 .

Theorem 4.9. *Let $\mathcal{S}$ be a collection of points in general position and D be any effective divisor on $X_{s,(0)}$. Assume that one of the following holds:*

(1) $s \leq n+2$,
(2) $s \geq n+3$ and $\sum_{i=1}^{n} m_i - nd \leq s-n-3$.

Then Question 1.11 has affirmative answer.

Proof. Since all cycles of the base locus of the divisors D are linear by Theorem 4.6, we conclude that $\widetilde{\mathcal{D}}$ equals $\tilde{D}$. The claims follow from Theorem 1.10. □

4.2. **Log resolutions for divisors on blown-up projective spaces in points.** Let D be any divisor on $X_{s,(0)}$ in the hypothesis of Theorem 4.9 and let $\bar{r}$ be the maximum dimension of its linear base locus. If the divisor D has $\bar{r} < n-1$, Theorem 4.9 implies that the map $X_{s,(\bar{r})} \to X_{s,(0)}$ is a resolution of singularities of D.

Moreover, we recall that a *log resolution* of the pair $(X_{s,(0)}, D)$ is a birational morphism $\pi : Y \to X_{s,(0)}$ such that the pair $(Y, \tilde{D})$ is *log smooth*, where $\tilde{D} = \pi_*^{-1} D$ is the strict transform of D, namely such that Y is smooth and the sum $\tilde{D} + \mathrm{Exc}(\pi)$, where $\mathrm{Exc}(\pi)$ is the sum of exceptional divisors of π, is simple normal crossing.

For D a general divisor of the linear system $|dH - \sum_{i=1}^{s} m_i E_i|$ as in Notation 1.1, we obtain the following result.

Corollary 4.10. *If D satisfies relation (4.1), then the pair $(X_{s,(\bar{r})}, D_{(\bar{r})})$ is log smooth and the birational morphism $X_{s,(\bar{r})} \to X_{s,(0)}$ is a* log resolution *of the singularities of the pair $(X_{s,(0)}, D)$.*

Proof. The variety $X_{s,(\bar{r})}$ is smooth. Indeed at each step of the blow-up of X, $X_{s,(r)} \to X_{s,(r-1)}$, $r \leq \bar{r}$, the center of the blow-up is a disjoint union of smooth subvarieties, namely the strict transforms of the liner cycles $L_{I(r)}$ with $k_{I(r)} > 0$ (see [24, Subsection 4.2] for details). This, together with the fact that each exceptional divisor is smooth, also proves that the sum $\mathrm{Exc}(\pi) = \sum_{r=1}^{n-2} \sum_{I(r)} E_{I(r)}$ is simple normal crossing. Moreover $D_{(\bar{r})}$ is base locus free and its support intersects transversally all exceptional divisors $E_{I(r)}$, by Theorem 4.6. Finally since D is general, then $D_{(\bar{r})}$ is general too hence it is smooth by Bertini's theorem.

□

5. Log abundance for pairs on blow-ups of $\mathbb{P}^n$

In this section we construct an infinte family of log canonical pairs given by effective divisors on the blow-up of $\mathbb{P}^n$ at s points in general position, $X_{s,(0)}$, with arbitrary s. Moreover we prove that the (log) abundance conjecture holds for these pairs.

In the case $s \leq n+3$, the blown-up space $X_{s,(0)}$ is log Fano (see [17, 41] and the recent preprint [2] for an explicit proof), hence a Mori Dream Space. Therefore it is a well-known fact that all nef divisors are semi-ample. Moreover this statement hold for $s \leq 2n$, see Theorem 3.2 and [2, Proposition 1.4]. Our contribution in this section is to provide an explicit constructive proof of the abundance conjecture for all effective divisors (Section 5.4).

For $s \geq n+4$, the result is new (Section 5.3). This implies that it is possible to run the Minimal Model Program and find good minimal models.

5.1. **Preliminary definitions.** For an introduction to singularities in the Minimal Model Program and in particular to the abundance conjecture, we refer for instance to [38]. Let (X,Δ) be a log pair, with X a normal variety and $\Delta=\sum_j a_j\Delta_j$ a formal $\mathbb{Q}$-linear combination of prime divisors. Let $\pi:Y\to X$ be a log resolution of (X,Δ), denote by $\tilde{\Delta}:=\pi_*^{-1}\Delta$ the strict transform of Δ and by
E_i the exceptional divisors. Write

$$K_Y+\tilde{\Delta}=\pi^*(K_X+\Delta)+\sum_i a(E_i,X,\Delta)E_i,$$

with $a(E_i,X,\Delta)\in\mathbb{Q}$. By [38, Corollary 2.32], if $a_j\leq 1$, then the *discrepancy* of the pair (X,Δ) can be computed as

$$\operatorname{discrep}(X,\Delta)=\min_i\left\{a(E_i,X,\Delta),\min_j\{1-a_j\},1\right\}. \tag{5.1}$$

The pair (X,Δ) is said to be *log canonical* (*lc*) if $\operatorname{discrep}(X,\Delta)\geq -1$.

Conjecture 5.1 ([38, Conjecture 3.12]). *Let (X,Δ) be lc, Δ effective. Then $K_X+\Delta$ is nef if and only if it is semi-ample.*

The abundance conjecture is known to hold for surfaces and threefolds (see [35]) and for fourfolds with positive Kodaira dimension (see [26]). Very little is known in higher dimension.

5.2. **Abundance theorem for blown-up projective spaces.** As an application of the results of this paper, we prove that the log abundance conjecture holds for $(X_{s,(0)},\Delta)$, where $\Delta:=\epsilon D$ with $0\leq\epsilon\leq 1$ and $D\geq 0$ effective divisor on $X_{s,(0)}$, by explicitly constructing a log resolution of the pair.

We recall here that effective divisors on $X_{s,(0)}$, with $s\leq n+3$ were classified in [14], see also Theorem 1.7. Moreover for arbitrary s, if (4.1) holds, the divisor D is effective, by [12, Theorem 5.3].

For a small number of points, $s\leq n+3$, or an arbitrary number of points with a bound of the coefficients of the divisors, we construct an infinite family of log canonical pairs given by effective divisors D on $X_{s,(0)}$.

Theorem 5.2. *Fix integers $n>3$, arbitrary s, and $\epsilon\in\mathbb{Q}$, with $0\leq\epsilon\ll 1$. Let $D=dH-\sum_{i=1}^s m_iE_i$ be a general effective divisor on $X=X_{s,(0)}$ with $s\leq n+3$ or with $s\geq n+4$ and satisfying (4.1), i.e.*

$$\sum_{i=1}^s m_i-nd\leq\max\{0,s-n-3\}.$$

Assume that

$$\begin{aligned}&\epsilon m_i\geq n-1,\ \forall i\in\{1,\dots,s\}\\&\epsilon(m_i+m_j-d)\leq n-3,\ \forall i,j\in\{1,\dots,s\},\ i\neq j.\end{aligned} \tag{5.2}$$

Then the pair (X,Δ) is lc.

Recall the classes of the canonical divisors on X,

$$K_X=-(n+1)H+(n-1)\sum E_i,$$

and consider the $\mathbb{Q}$-divisor

$$K_X+\Delta=(\epsilon d-n-1)H-\sum_i(\epsilon m_i-n+1)E_i.$$

Proposition 5.3. *In the same hypothesis of Theorem 5.2, Conjecture 5.1 holds for the pair (X, Δ), namely if $K_X + \Delta$ is nef then it is semi-ample.*

Proof. By Theorem 5.2, the pair (X, Δ) is lc. To conclude, it is easy to see that under the condition (4.1), then the divisor $K_X + \Delta$ is nef (equiv. semi-ample) if and only if the conditions (5.2) of Theorem 5.2 are verified. This follows from Theorem 3.1. In fact, if $s \leq 2n$, the nef and the semi-ample cone coincide, and the thesis follows trivially. Otherwise, if $s \geq 2n + 1$, we just have to check that (4.1) implies condition (2.2) of of Theorem 3.1 (and of Theorem 2.2) with $l = 0$. We leave the details of this to the reader. □

Corollary 5.4. *In the same notation as Theorem 5.2, the canonical ring*

$$\bigoplus_{l \geq 0} H^0(X, \mathcal{O}_X(lK_X + \lfloor l\Delta \rfloor))$$

is finitely generated.

Proof. See [38, Section 3.13]. □

Recall that in the space $\mathrm{N}^1(X_{s,(0)})_{\mathbb{R}}$, the Néron-Severi group tensored with the real numbers, the semi-ample divisors on $X_{s,(0)}$ are described in Theorem 3.1 respectively. Also the notion of stable base locus of line bundles, that in our case corresponds to that of base locus, can be extended to the case of $\mathbb{R}$-divisors by taking real multiples of line bundles, see Remark 4.8. Hence we can extend Proposition 5.3 to $\mathbb{R}$-divisors.

Proposition 5.5. *Let $0 \leq \epsilon \ll 1$, $\epsilon \in \mathbb{R}$. Let $\Delta = \epsilon D \in N^1(X_{s,(0)})_{\mathbb{R}}$ be an $\mathbb{R}$-divisor on $X_{s,(0)}$ satisfying the assumption of Theorem 5.2. Then Conjecture 5.1 holds for the pair (X, Δ).*

Remark 5.6. In the notation of Theorem 5.2, if $s \leq n+2$ or $s \geq n+3$ and condition (4.1) is satisfied, then D is effective and *only linearly obstructed*, see [12, Theorem 5.3]. In this case a log resolution of the corresponding pair is given by the iterated blow-up of the linear base locus, see Corollary 4.10.

The case $s = n + 3$ is the first case where non-linear obstructions appear for divisors violating (4.1). The base locus of effective divisors D was studied in [14] (cfr. Lemma 1.5). In this case a log resolution of any pair given satisfying (5.2), will be constructed in Section 5.4.1 by means of the iterated blow-up along the subvarieties, linear and non-linear, contained in the base locus.

5.3. **Proof of Theorem 5.2, only linearly obstructed case.** Let s be an arbitrary integer. Set $X := X_{s,(0)}$ and $Y := X_{s,(n-2)}$. Let

$$D = dH - \sum_{i=1}^{s} m_i E_i \geq 0$$

be a divisor as in Notation 1.1.

Assume moreover that D is only linearly obstructed, namely that condition (4.1) is verified.

Remark 5.7. Notice that under the assumption (5.2) the divisor D is irreducible. We recall that D represents a general member of the linear system $|D|$ and the assumptions above force D to have no divisorial components. Indeed, observe that

no (strict transform of) hyperplane spanned by n points is contained in the base locus of D. In fact, we can see first of all that (5.2) implies that for every $i = 1 \dots s$,

$$\epsilon(m_i - d) \leq -2. \tag{5.3}$$

Now, if $I = I(n-1)$ is an index set of cardinality n, we can compute that the multiplicity of containment of the corresponding hyperplane is zero as follows:

$$\begin{aligned}\sum_{i \in I} m_i - (n-1)d &= \left(\sum_{i \in I \setminus \{i_1, i_2\}} m_i - (n-2)d \right) + (m_{i_1} + m_{i_2} - d) \\ &\leq -2(n-2) + (n-3) \\ &\leq 0.\end{aligned}$$

Using the notation introduced in Subsection 1.3, let $\pi : Y \to X$ be the composition of blow-ups of X along lines, then planes etc., up to codimension-2 linear cycles L_I, $I \subset \{1, \dots, s\}$ for which $k_I > 0$, see (1.4) for the definition of k_I. As in (1.13), the strict transform of D, is given by

$$\tilde{D} = dH - \sum_i m_i E_i - \sum_{r=1}^{n-2} \sum_{I(r)} k_{I(r)} E_{I(r)}.$$

Proposition 5.8. *In the above notation, the map $\pi : Y \to X$ is a log resolution of (X, Δ), for every $\epsilon \geq 0$.*

Proof. It follows from Corollary 4.10.

□

Recall the classes of the canonical divisors on Y:

$$K_Y = -(n+1)H + (n-1) \sum E_i + \sum_{r=1}^{n-2} (n-r-1) \sum_{I(r)} E_{I(r)}.$$

Here we abuse notation by denoting by H the hyperplane class in both X and Y; similarly by abuse of notation we denote by E_i an exceptional divisor in X and its strict transform in Y.

For $0 \leq \epsilon < 1$, $\epsilon \in \mathbb{Q}$, consider the pairs $(X, \Delta) = (X_{s,(0)}, \epsilon D)$ and $(Y, \tilde{\Delta}) = (X_{s,(n-2)}, \epsilon \tilde{D})$ and write

$$K_Y + \tilde{\Delta} = \pi^*(K_X + \Delta) + \sum_{1 \leq r \leq n-2} (n - r - 1 - \epsilon k_{I(r)}) E_{I(r)}.$$

We have

$$a_i(E_{I(r)}, X, \Delta) = n - r - 1 - \epsilon k_{I(r)},$$

for any $I(r)$ such that $1 \leq r \leq n-2$ (cfr. [38, Lemma 2.29]). Therefore $\operatorname{discrep}(X, \Delta) \geq -1$ if

$$\epsilon k_{I(r)} \leq n - r, \ \forall I(r), \ 1 \leq r \leq n-2. \tag{5.4}$$

We are now ready to prove the first main result of this section.

Proof of Theorem 5.2, only linearly obstructed case. We are now ready to prove Theorem 5.2 for divisors with only linear obstructions. By Proposition 5.8, $(Y, \tilde{\Delta})$ is log smooth and $\pi : Y \to X$ is a log resolution of (X, Δ). We are going to prove that the pair is (X, Δ) is lc.

We prove that (5.4) is implied by (5.2), second line, and by (5.3). Indeed, take for instance $r = 2$. If $k_{I(2)} = 0$ the statement is obvious. Assume that $k_{I(2)} > 0$. Write $I(2) = \{i_1, i_2, i_3\}$. We have $\epsilon k_{I(2)} = \epsilon((m_{i_1} - d) + (m_{i_2} + m_{i_3} - d)) \leq -2 + (n-3) \leq n - 2$. The same holds for $r \geq 3$. □

5.4. **Proof of Theorem 5.2, case $s = n+3$.** In this section we conclude the proof of Theorem 5.2. In particular we consider all effective divisors with $s = n + 3$ with no restriction on the obstructions contained in the base locus and we construct a log resolution of the corresponding pairs.

Take $X := X_{n+3,(0)}$ and a divisor D on X of the form

$$D = dH - \sum_{i=1}^{n+3} m_i E_i \geq 0.$$

Write $\Delta = \epsilon D$, for $0 \leq \epsilon \ll 1$ and assume it satisfies condition (5.2) of Theorem 5.2.

Remark 5.9. Note that, (5.2) implies that D is big (cfr. Theorem 3.5.). In fact we can use (5.8) that gives

$$\epsilon k_{I(n-2t),\sigma_t} \leq -n - 1 < 0,$$

for all $I(n - 2t),\ t \geq 1$. Moreover one can similarly check that

$$\epsilon k_{I(n-2t-1),\sigma_t} \leq 0,$$

for all $I(n - 2t - 1),\ t \geq 1$, hence D is movable, see Theorem 1.7. In particular the general element of $|D|$ is irreducible.

Remark 5.10. The argument developed in Section 5.3 applies also to the case $s = n + 3$ and $k_C := k_C(D) = 0$ (see (1.6) for the definition of this number), namely to divisors D on $X_{n+3,(0)}$ that have only linear base locus and for which (4.1) is satisfied. Indeed under this condition and (5.2), D is movable (cfr. Theorem 1.11) and its strict transform on $X_{n+3,(n-2)}$, the iterated blow-up along the linear cycles, is globally generated by Theorem 4.1. Moreover the lc condition on the pair $(X, \epsilon\Delta)$ can be verified in the same manner as in the case with (4.1), as in Section 5.3.

From now on we will assume that $k_C(D) \geq 1$.

5.4.1. *Constructing a log resolution of* $(X, \epsilon D)$. The following result computes the pairwise intersections of the joins $\mathrm{J}(L_I, \sigma_t)$.

Proposition 5.11. *Let* $\mathrm{J}(L_{I_1}, \sigma_{t_1})$ *and* $\mathrm{J}(L_{I_2}, \sigma_{t_2})$ *be any join varieties of the same dimension* $r_{I_1,\sigma_{t_1}} = r_{I_2,\sigma_{t_2}}$*. Then,*

$$\mathrm{J}(L_{I_1}, \sigma_{t_1}) \cap \mathrm{J}(L_{I_2}, \sigma_{t_2}) = \bigcup_J \mathrm{J}(L_J, \sigma_{t_J}),$$

for some subsets $J \subseteq I_1 \cup I_2$.

Proof. Assume first of all that $I_1 \cap I_2 = \emptyset$. We have

$$0 \leq 2(n-1) - 2\left(r_{I_1,\sigma_{t_1}} + r_{I_2,\sigma_{t_2}}\right),$$

which gives

$$2r_{I_i,\sigma_{t_i}} \leq 2n - (r_{I_1,\sigma_{t_1}} + 1) - (r_{I_2,\sigma_{t_2}} + 1),\ \forall i \in \{1,2\}.$$

Therefore, since $|I_i| \leq r_{I_i,\sigma_{t_i}} + 1$, we have $2r_{I_i,\sigma_{t_i}} \leq 2n - (|I_1| + |I_2|)$, $i = 1,2$, hence we are in the hypotheses of Proposition 1.13. This concludes the proof of the statement in the case when I_1, I_2 are disjoint. Indeed at each step of blow-up, the intersection of any two subvarieties that are being blown-up is a union of smaller subvarieties that have been previously blown-up.

Assume now that $I_1 \cap I_2 =: I_{12} \neq \emptyset$. Set $I_i' := I_i \setminus I_{12}$, for $i = 1,2$. Notice that

$$r_{I_i',\sigma_{t_i}} = r_{I_i,\sigma_{t_i}} - |I_{12}|,\ \forall i \in \{1,2\}.$$

Moreover,

$$\mathrm{J}(\mathrm{J}(L_{I_i'},\sigma_{t_i}), L_{I_{12}}) = \mathrm{J}(L_{I_i},\sigma_{t_i}),\ \forall i \in \{1,2\}. \tag{5.5}$$

Also,

$$\mathrm{J}(\mathrm{J}(L_{I_1'},\sigma_{t_1}), L_{I_{12}}) \cap \mathrm{J}(\mathrm{J}(L_{I_2'},\sigma_{t_2}), L_{I_{12}}) = \mathrm{J}(\mathrm{J}(L_{I_1'},\sigma_{t_1}) \cap \mathrm{J}(L_{I_2'},\sigma_{t_2}), L_{I_{12}}).$$

One proves the two last equalities of subvarieties using induction on $|I_{12}|$ based on the case $|I_{12}| = 0$ for which the statement is obvious. With a similar argument as the one employed in the first case, we obtain that

$$\begin{aligned} 2r_{I_i',\sigma_{t_i}} &\leq 2n - (r_{I_1',\sigma_{t_1}} + 1) - (r_{I_2',\sigma_{t_2}} + 1) - 4|I_{12}|, \\ &\leq 2n - (|I_1'| + |I_2'|)), \end{aligned}$$

for $i = 1,2$. Therefore,

$$\mathrm{J}(L_{I_1},\sigma_{t_1}) \cap \mathrm{J}(L_{I_2},\sigma_{t_2}) = \mathrm{J}\left(\bigcup_J \mathrm{J}(L_J,\sigma_{t_J}), L_{I_{12}}\right) = \bigcup_J \mathrm{J}(L_{J\cup I_{12}},\sigma_{t_J}), \tag{5.6}$$

where the union is taken over all subsets $J \subseteq I_1' \cup I_2'$ satisfying $2|(I_i' \cup J) \setminus (I_i \cap J)| = |I_1'| + |I_2'|,\ \forall i \in \{1,2\}$, and for every such J, t_J is the integer defined by the following equation $2r_{J,\sigma_{t_J}} = 2r_{I_i',\sigma_{t_i}} - (|I_1'| + |I_2'|)$. The first equality of (5.6) follows from Proposition 1.13, the second inequality is an application of (5.5). $\square$

Notation 5.12. For every n, let $\mathcal{T} := \{(I,t) : 1 \leq r_{I,\sigma_t} \leq n-2\}$ be the set parametrizing all subvarieties $\mathrm{J}(L_I,\sigma_t)$ of $\mathbb{P}^n$ of dimension between 1 and $n-2$.

(1) If n is even, say $n = 2\nu$, with $\nu \geq 2$, we consider the following set

$$\mathcal{C}^{\text{even}} := \{(I,t) \in \mathcal{T} : 2r_{I,\sigma_t} \leq 2\nu - 1 \text{ if } I \neq \emptyset\}.$$

In other terms, $(I,t) \in \mathcal{C}^{\text{even}}$ if $I = \emptyset$ and $1 \leq t < \nu$ or if $I \neq \emptyset$ and $|I| + 2t \leq \nu$.

(2) If n is odd, say $n = 2\nu + 1$, with $\nu \geq 2$, we consider the following set

$$\mathcal{C}^{\text{odd}} := \{(I,t) \in \mathcal{T} : 2r_{I,\sigma_t} \leq 2\nu \text{ if } I \neq \emptyset, \{1\}\}.$$

In other terms, $(I,t) \in \mathcal{C}^{\text{even}}$ if $I = \emptyset, \{1\}$ and $1 \leq t < \nu$ or if $I \neq \emptyset, \{1\}$ and $|I| + 2t \leq \nu + 1$.

Using the same notation as in Subsection 1.4, we consider

$$\pi^\sigma : Y^\sigma \to X,$$

the iterated blow-up of $X = X_{2\nu+3,(0)}$ along all subvarieties $\mathrm{J}(L_I, \sigma_t)$ such that $(I,t) \in \mathcal{C}^{\mathrm{even}}$ if n is even, and such that $(I,t) \in \mathcal{C}^{\mathrm{odd}}$ if n is odd, in increasing dimension. We denote by E_{I,σ_t} the corresponding exceptional divisors.

Let us denote $\mathrm{Exc}(\pi^\sigma) = \sum_{(I,t)\in\mathcal{C}^{\mathrm{even}}} E_{I,\sigma_t}$ if n is even and $\mathrm{Exc}(\pi^\sigma) = \sum_{(I,t)\in\mathcal{C}^{\mathrm{odd}}} E_{I,\sigma_t}$ if n is odd.

Proposition 5.13. *For any n, the blown-up space Y^σ is smooth. Moreover* $\mathrm{Exc}(\pi^\sigma)$ *is simple normal crossing.*

Proof. At each level of the blow-up, the center is a disjoint union of smooth subvarieties. Indeed, if $\mathrm{J}(L_{I_1}, \sigma_{t_1})$ and $\mathrm{J}(L_{I_2}, \sigma_{t_2})$ are two subvarieties such that $2r_{I_1,\sigma_{t_1}} = 2r_{I_2,\sigma_{t_2}} \le n-1$, then we conclude by Proposition 5.11. Otherwise, if $2r_{I,\sigma_t} \ge n$, the only subvarieties that we blow-up are the strict transforms of the secant varieties σ_t's, with $\frac{\nu+1}{2} \le t \le \nu$, in the case $n = 2\nu$, and the strict transforms of the σ_t's and of the pointed cones $\mathrm{J}(L_{\{1\}}, \sigma_t)$, with $\frac{\nu}{2}+1 \le t \le \nu$, if $n = 2\nu+1$. It is easy to see that such pointed cones intersect along smaller joins of the same form, that have been previously blown-up. □

5.4.2. *Properties of the strict transform of $|\epsilon D|$.* Denote by $\tilde{D}^\sigma$ the strict transform of D, as in (1.17):

$$\tilde{D}^\sigma := dH - \sum_i m_i E_i - \sum_{r=1}^{n-2} \sum_{\substack{I,t:\\ r_{I,\sigma_t}=r}} k_{I,\sigma_t} E_{I,\sigma_t}. \tag{5.7}$$

Remark 5.14. Notice that, as in the only linearly obstructed case (handled in Section 5.3), the integer r in the above summation of exceptional divisors ranges up to $n-2$, as D is movable (cfr. Theorem 1.7).

We use the sets $\mathcal{C}^{\mathrm{even}}$ and $\mathcal{C}^{\mathrm{odd}}$ introduced in Notation 5.12 in the following lemma.

Lemma 5.15. *For any n, $s = n+3$ and D as in Theorem 5.2, if $k_{I,\sigma_t} > 0$, then $2r_{I,\sigma_t} < n-1$. In particular such a pair $(I,t) \in \mathcal{T}$ belongs to $\mathcal{C}^{even}$ if n is even, and to $\mathcal{C}^{odd}$, if n is odd.*

Proof. Let us compute

$$\begin{aligned}
\epsilon k_C &= \epsilon \max\left\{0, \left(\sum_{i=1}^{n-3} m_i - (n-3)d\right) + (m_{n-2} + m_{n-1} - d) + \right.\\
&\qquad \left. + (m_n + m_{n+1} - d) + (m_{n+2} + m_{n+3} - d)\right\}\\
&\le -2(n-3) + 3(n-2)\\
&= n-3,
\end{aligned}$$

where $k_C := k_{\sigma_1}$ is the multiplicity of containment of the rational normal curve of degree n. If $k_C > 0$, we also compute

$$\begin{aligned}\epsilon(k_C - d) &= \epsilon\left(\sum_{i=1}^{n-1} m_i - (n-1)d + (m_n + m_{n+1} - d) + (m_{n+2} + m_{n+3} - d)\right)\\ &\leq -2(n-1) + 2(n-2)\\ &= -4.\end{aligned}$$

Both of the above inequalities follow from (5.2), second line, and the inequality (5.3), computed in the proof of Theorem 5.2. Hence, for every $t \geq 0$, we obtain

$$\begin{aligned}\epsilon k_{\sigma_t} &= \epsilon \max\{0, k_C + (t-1)(k_C - d)\}\\ &\leq \max\{0, n-3-4(t-1)\}.\end{aligned}$$

Furthermore, for every I with $|I| \geq 0$ and $t \geq 0$, we obtain

$$\begin{aligned}\epsilon k_{I,\sigma_t} &= \epsilon \max\left\{0, k_{\sigma_t} + \sum_{i\in I} m_i - (|I|)d\right\}\\ &\leq \max\{0, (n-3-4(t-1)) - 2|I|\}\\ &= \max\{0, (n-|I|-2t+1) - (|I|+2t)\}.\end{aligned}$$

Hence, for every variety $\mathrm{J}(L_I, \sigma_t) \subset \mathbb{P}^n$ with $r_{I,\sigma_t} = |I| + 2t - 1 \leq n$, we obtain

$$\epsilon k_{I,\sigma_t} \leq \max\{0, n-1-2r_{I,\sigma_t}\}. \tag{5.8}$$

□

Corollary 5.16. *In the same notation as Lemma 5.15, the divisor $\tilde{D}^\sigma$ intersects transversally each exceptional divisors.*

Theorem 5.17. *For any n, $s = n+3$ and D as in Theorem 5.2. If the divisor D is general, then $\tilde{D}^\sigma$ is base point free.*

We will show Theorem 5.17 in Section 5.4.3 (if n is even) and in Section 5.4.4 (is n is odd). The next corollary follows from Bertini's Theorem.

Corollary 5.18. *In the notation of Theorem 5.17, the general element of $|\tilde{D}^\sigma|$ is smooth.*

5.4.3. *Proof of Theorem 5.17, case n even.* Write $n = 2\nu$, $\nu \geq 2$. Recall that the strict transform on $X_{2\nu+3,(0)}$ of the secant variety $\sigma_\nu \subset \mathbb{P}^{2\nu}$ is the fixed divisor

$$\Sigma := (\nu+1)H - \nu \sum_{i=1}^{2\nu+3} E_i.$$

Proposition 5.19. *In the above notation, the strict transform $\tilde{\Sigma}^\sigma$ of Σ on Y^σ is smooth.*

Proof. Recall that the singular locus of $\sigma_\nu \subset \mathbb{P}^{2\nu}$ is $\sigma_{\nu-1} \subset \sigma_\nu$ and, more precisely, the non-reduced union of all joins $\mathrm{J}(L_I, \sigma_t)$ such that $k_{I,\sigma_t}(\Sigma) > 1$. In particular we compute the following multiplicities. For every I and t such that $|I|, t \geq 0$, we have

$$k_{I,\sigma_t}(\Sigma) = \max\{0, \nu - |I| - t + 1\}. \tag{5.9}$$

All subvarieties $\mathrm{J}(L_I, \sigma_t)$ such that $|I| + t \leq \nu$ have been blown-up, hence we conclude that π^σ is a resolution of the singularities of Σ. $\square$

Choose $\alpha \in \mathbb{N}$ such that

$$\frac{k_C(D)}{\nu} \leq \alpha \leq \min_{1 \leq i \leq 2\nu+3} \left\{ \frac{m_i}{\nu}, d - m_i \right\}. \tag{5.10}$$

Lemma 5.20. *Under the assumptions of Theorem 5.2, such an integer α exists.*

Proof. It is enough to prove that $\frac{k_C(D)}{\nu} \leq \frac{m_i}{\nu} - 1$ and that $\frac{k_C(D)}{\nu} \leq d - m_i - 1$ for all $i \in \{1 \dots, 2\nu + 3\}$. The first statement follows from the following computation:

$$\epsilon k_C(D) \leq 2\nu - 3 \leq 2\nu - 1 - \epsilon\nu \leq \epsilon(m_i - \nu).$$

The first inequality follows from (5.8). The second inequality follows from the assumption $\epsilon \ll 1$; in fact it is enough to take $\epsilon \leq \frac{4}{n} = \frac{2}{\nu}$. The last inequality follows from (5.2). Similarly, one proves the second statement by observing that

$$\epsilon k_C(D) \leq 2\nu - 3 \leq 2\nu - \epsilon\nu \leq \epsilon\nu(d - m_i - 1).$$

The last inequality follows from (5.3). $\square$

Consider the linear system

$$|D'| := |D - \alpha\Sigma| = |d'H - \sum_{i=1}^{2\nu+3} m'_i E_i|,$$

with $d' := d - \alpha(\nu + 1)$ and $m'_i := m_i - \alpha\nu$, for all $i \in \{1 \dots, 2\nu + 3\}$. We have

$$|D'| + \alpha\Sigma \subseteq |D|.$$

Proposition 5.21. *In the notation of above, the linear system $|D'|$ is non-empty. Moreover $|D'|$ has only linear base locus and the strict transform in Y^σ of $|D'|$ is base point free.*

Proof. For every $i \in \{1 \dots, 2\nu+3\}$, since by (5.10) we have $\alpha \leq \frac{m_i}{\nu}$ and $\alpha \leq d - m_i$, then $m'_i \geq 0$ and $d \geq m'_i$ respectively. Moreover, let us compute

$$\begin{aligned} k_C(D') &= \max\left\{0, \sum_{i=1}^{2\nu+3} (m_i - \alpha\nu) - 2\nu(d - \alpha(\nu + 1))\right\} \\ &= \max\{0, k_C(D) - \alpha\nu\} \\ &= 0. \end{aligned}$$

The last equality follows from (5.10): $\alpha \geq \frac{k_C(D)}{\nu}$. This proves the first statement, namely that Δ' is effective, see Theorem 1.7.

To prove the second statement, notice that if $|I| \leq \nu$, then the pair $(I, 0) \in \mathcal{C}^{\text{even}}$, hence the corresponding linear subspace $L_I \subset \mathbb{P}^{2\nu}$ has been blown-up. Otherwise, if $|I| \geq \nu + 1$, we claim that $k_I(D') = 0$. Theorem 4.1 imply the second statement, namely that $\tilde{D}^\sigma$ is a globally generated divisor. To prove the claim, for every I we

choose $J \subset I$ with $|J| = \nu$ and we compute

$$\begin{aligned}\epsilon k_I(D) &= \max\left\{0, \epsilon\left(\sum_{i\in I} m_i - (|I|-1)d\right) + \epsilon\alpha(|I|-\nu-1)\right\}\\ &= \max\left\{0, \epsilon\left(\sum_{i\in J} m_i - (\nu-1)d\right) + \epsilon\left(\sum_{i\in I\setminus J}(m_i-d)\right) + \epsilon\alpha(|I|-\nu-1)\right\}\\ &\leq \max\{0, -\epsilon\alpha\}\\ &= 0.\end{aligned}$$

The inequality holds because $\epsilon\left(\sum_{i\in J} m_i - (\nu-1)d\right) \leq 0$ by (5.8) and $\alpha(|I|-\nu) \leq \sum_{i\in I\setminus J}(d-m_i)$ by (5.10). □

Proposition 5.22. *In the notation of above, there exist non-negative numbers $\alpha_{I,\sigma_t} \in \mathbb{Z}$ such that*

$$|\tilde{D'}^{\sigma}| + \alpha\tilde{\Sigma}^{\sigma} + \sum_{(I,t)\in\mathcal{C}^{even}} \alpha_{I,\sigma_t} E_{I,\sigma_t} \subseteq |\tilde{D}^{\sigma}|.$$

Proof. In order to prove the statement, we compare the strict transform, $|\tilde{D'}^{\sigma}|+\alpha\tilde{\Sigma}^{\sigma}$ with that the strict transform of D.

Recall that, for any $1 \leq t < \nu$ and I such that $|I| \geq 0$, we have $k_{I,\sigma_t}(D') = 0$ by Proposition 5.21. Moreover for all I and t, (5.8) implies that

$$k_{I,\sigma_t}(\alpha\Sigma) = \alpha k_{I,\sigma_t}(\Sigma) \geq k_{I,\sigma_t}(D). \tag{5.11}$$

This, together with (5.9) and the computation of $k_I(D)$ made in the proof of Proposition 5.21, implies the following linear equivalence of divisors

$$\tilde{D}^{\sigma} \sim \tilde{D'}^{\sigma} + \alpha\tilde{\Sigma}^{\sigma} + \sum_{(I,t)\in\mathcal{C}^{\text{even}}} \alpha_{I,\sigma_t} E_{I,\sigma_t},$$

where $\alpha_I := k_I(\alpha\Sigma) + k_I(D') - k_I(D)$ if $1 \leq |I| \leq \nu$, while $\alpha_{I,\sigma_t} := k_{I,\sigma_t}(\alpha\Sigma) - k_{I,\sigma_t}(D)$ for $1 \leq t \leq \nu$, $\nu+1 \leq |I| \leq 2\nu$. Finally, from (5.11) we obtain $a_{I,\sigma_t} \geq 0$, for all I and t. □

Proof of Theorem 5.17, case n even. We recall that a general member of $|D|$ vanishes at every point of $J(L_I, \sigma_t)$ with multiplicity equal to k_{I,σ_t} and that $|\tilde{D}^{\sigma}|$ is the strict transform of D under the blow-up of its secant base locus. Assume that the base locus, $\mathrm{Bs}|\tilde{D}^{\sigma}|$, is non-empty. By Proposition 5.22, we have

$$\mathrm{Bs}|\tilde{D}^{\sigma}| \subset \mathrm{Bs}|\tilde{D'}^{\sigma}| \cup \tilde{\Sigma} \cup \bigcup_{(I,t)\in\mathcal{C}^{\text{even}}} E_{I,\sigma_t}.$$

Note that the divisor $|\tilde{D'}^{\sigma}|$ is base point free by Theorem 4.1, therefore $\mathrm{Bs}|\tilde{D'}^{\sigma}|$ is empty. Hence

$$\mathrm{Bs}|\tilde{D}^{\sigma}| \subset \tilde{\Sigma} \cup \bigcup_{(I,t)\in\mathcal{C}^{\text{even}}} E_{I,\sigma_t}.$$

Assume there is a base point for $|\tilde{D}^{\sigma}|$ on one of the exceptional divisors E_{I,σ_t}, $(I,t) \in \mathcal{C}^{\text{even}}$, or on $\tilde{\Sigma}$. This implies the existence of a point in a join $J(L_I, \sigma_t)$, or in Σ, on which the general divisor D has infinitesimal tangencies. The proof of Proposition 1.6 implies that no point of $J(L_I, \sigma_t)$, nor of Σ, carries any infinitesimal information. This leads to a contradiction.

□

5.4.4. *Proof of Theorem 5.17, case n odd.* Recall that the strict transform on $X_{2\nu+3,(0)}$ of the cone $\mathrm{J}(L_{\{1\}},\sigma_\nu)\subset\mathbb{P}^{2\nu}$ is the divisor

$$\Gamma := (\nu+1)H-(\nu+1)E_1-\nu\sum_{i=2}^{2\nu+4}E_i.$$

Proposition 5.23. *In the above notation, the strict transform $\tilde{\Gamma}^\sigma$ of Γ on Y^σ is smooth.*

Proof. Recall that the singular locus of $\mathrm{J}(L_{\{1\}},\sigma_\nu)\subset\mathbb{P}^{2\nu}$ is $\mathrm{J}(L_{\{1\}},\sigma_{\nu-1})\subset\mathrm{J}(L_{\{1\}},\sigma_\nu)$ and, more precisely, the non-reduced union of all joins $\mathrm{J}(L_I,\sigma_t)$ such that $k_{I,\sigma_t}(\Sigma)>1$. In particular we compute the following multiplicities. For every I and t such that $|I|,t\geq 0$, we have

$$k_{I,\sigma_t}(\Sigma)=\max\{0,\nu-|I|-t+1+\delta_I\}, \tag{5.12}$$

where δ_I is defined as

$$\delta_I:=\delta_{1,I}=\begin{cases}1 & \text{if } 1\in I\\ 0 & \text{if } 1\notin I.\end{cases} \tag{5.13}$$

All subvarieties $\mathrm{J}(L_I,\sigma_t)$ such that $|I|+t\leq\nu+\delta_I$ have been blown-up, hence we conclude that π^σ is a resolution of the singularities of Σ. □

Choose $\beta\in\mathbb{N}$ such that

$$\frac{k_C(D)}{\nu}\leq\beta\leq\min_{1\leq i\leq 2\nu+4}\left\{\frac{m_1}{\nu+\delta_{\{i\}}},d-m_i\right\}. \tag{5.14}$$

where $\delta_{\{i\}}$ is the Kronecker delta defined in (5.13)

Lemma 5.24. *Under the assumptions of Theorem 5.2, such an integer β exists.*

Proof. For every $i\in\{1,\ldots,2\nu+4\}$, we compute

$$\epsilon k_C(D)\leq 2\nu-2\leq 2\frac{\nu^2}{\nu+\delta_{\{i\}}}-\epsilon\nu\leq\epsilon\nu\left(\frac{m_1}{\nu+\delta_{\{i\}}}-1\right).$$

In the above expression, the first inequality follows from (5.8). The second inequality follows from the assumption $\epsilon\ll 1$; in fact it is enough to take $\epsilon\leq\frac{8}{n^2-1}=\frac{2}{\nu^2+\nu}$. The last inequality follows from (5.2). Furthermore we have

$$\epsilon k_C(D)\leq 2\nu-2\leq 2\nu-\epsilon\nu\leq\epsilon\nu(d-m_i-1).$$

The last inequality follows from (5.3). □

Consider the linear system

$$|D'|:=|D-\beta\Gamma|=|d'H-\sum_{i=1}^{2\nu+4}m_i'E_i|,$$

with $d':=d-\beta(\nu+1)$, $m_1':=m_1-\beta(\nu+1)$ and $m_i':=m_i-\beta\nu$, for all $i\in\{2\ldots,2\nu+4\}$. We have

$$|D'|+\beta\Gamma\subseteq|D|.$$

Proposition 5.25. *In the notation of above, the linear system $|D'|$ is non-empty. Moreover $|D'|$ has only linear base locus and the strict transform in Y^σ of $|D'|$ is base point free.*

Proof. By (5.14), we have $0 \leq m'_i \leq d$ for all $i \in \{1, \dots, 2\nu+4\}$. Moreover, as in the proof of Proposition 5.21, we compute

$$k_C(D') = \max\left\{0, \epsilon\left(\sum_{i=1}^{2\nu+4} m'_i - (2\nu+1)d'\right)\right\} = \max\{0, k_C(D) - \beta\nu\} = 0.$$

This proves the first statement.

To prove the second statement, we notice that if $|I| \leq \nu+1$ the pair $(I, 0) \in \mathcal{C}^{\text{odd}}$, while we claim that $k_I(D') = 0$ if $|I| \geq \nu+2$. This and Theorem 4.1 imply the second statement. We prove the claim for $1 \in I$; the case $1 \notin I$ is similar and we leave the details to the reader. Choose $J \subset I$ with $|J| = \nu+1$ and $1 \in J$. Let us compute

$$\begin{aligned}
\epsilon k_I(D) &= \max\left\{0, \epsilon\left(\sum_{i\in I} m_i - (|I|-1)d\right) + \epsilon\beta(|I| - \nu - 2)\right\}\\
&= \max\left\{0, \epsilon\left(\sum_{i\in J} m_i - (\nu)d\right) + \epsilon\left(\sum_{i\in i\setminus J}(m_i - d)\right) + \epsilon\beta(|I| - \nu - 2)\right\}\\
&\leq \max\{0, -\epsilon\beta\}\\
&= 0.
\end{aligned}$$

The inequality holds thanks to (5.8) and (5.14). ☐

Proposition 5.26. *In the notation of above, there exist non-negative numbers $\beta_{I,\sigma_t} \in \mathbb{Z}$ such that*

$$|\tilde{D'}^{\sigma}| + \beta\tilde{\Gamma}^{\sigma} + \sum_{(I,t)\in\mathcal{C}^{odd}} \beta_{I,\sigma_t} E_{I,\sigma_t} \subseteq |\tilde{D}^{\sigma}|.$$

Proof. The proof follows the same lines as that of Proposition 5.22 and it uses Proposition 5.25. We leave the details to the reader. ☐

Proof of Theorem 5.17, case n odd. The proof follows the same idea as that of the case n even, at the end of Section 5.4.3.

☐

5.4.5. *The pair $(X, \epsilon D)$ is lc, for D general.* Notice that the canonical divisor of Y^σ is

$$K_{Y^\sigma} = -(n+1)H + (n-1)\sum E_i + \sum_{r=1}^{n-2}(n-r-1)\sum_{\substack{I,t:\\ r_{I,\sigma_t}=r}} E_{I,\sigma_t}.$$

We are now ready to prove Theorem 5.2 for $s = n+3$.

Proof of Theorem 5.2, case $s = n+3$. By Corollary 5.18, $(Y^\sigma, \tilde{\Delta})$ is log smooth and $\pi : Y \to X$ is a log resolution of (X, Δ).

To complete the proof, similarly to the case of only linearly obstructed divisors, we are going to show that (5.2) implies

$$\epsilon k_{I,\sigma_t} \leq n - |I| - 2t + 1, \quad \forall I(r),\ 2 \leq r_{I,\sigma_t} \leq n-2, \tag{5.15}$$

that in turns implies that $\operatorname{discrep}(X, \Delta) \geq -1$.

This follows from the inequalities (5.8) computed in the proof of Proposition Proposition 5.15. ☐

6. On the F-conjecture for $\overline{\mathcal{M}}_{0,n}$

Let $\overline{\mathcal{M}}_{0,n}$ be the *moduli space of stable rational curves with* n *marked points*. For $n=5$, $\overline{\mathcal{M}}_{0,n}$ is a del Pezzo surface and it has the property of being a Mori dream space. Hu and Keel in [32] showed that $\overline{\mathcal{M}}_{0,6}$ is a log Fano threefold, hence a Mori Dream Space; Castravet computed its Cox ring in [16]. For $n\geq 13$, $\overline{\mathcal{M}}_{0,n}$ is known to not be a Mori Dream Space, see [18, 29].

We recall here the *F-conjecture* on the nef cone of $\overline{\mathcal{M}}_{0,n}$ due to Fulton. The elements of the 1-dimensional boundary strata on $\overline{\mathcal{M}}_{0,n}$ are called *F-curves*. A divisor intersecting non-negatively all F-curves is said to be *F-nef*. The F-Conjecture states that a divisor on $\overline{\mathcal{M}}_{0,n}$ is nef if and only if it is F-nef. This conjecture was proved for $n\leq 7$ in by Keel and McKernan [36].

6.1. Preliminaries and notation.

Let $\mathcal{I}$ be a subset of $\{1,\ldots,n+3\}$ with cardinality $2\leq|\mathcal{I}|\leq n+1$ and let $\Delta_{\mathcal{I}}$ denote a boundary divisor on $\overline{\mathcal{M}}_{0,n+3}$. Here, $\Delta_{\mathcal{I}}$ is the divisor parametrizing curves with one component marked by the elements of $\mathcal{I}$ and the other component marked by elements of its complement, $\mathcal{I}^c$, in $\{1,\ldots,n+3\}$. Obviously $\Delta_{\mathcal{I}}=\Delta_{\mathcal{I}^c}$.

We recall that, for any $1\leq i\leq n+3$, the tautological class ψ_i is defined as the first Chern class of the cotangent bundle, $c_1(\mathbb{L}_i)$, where $\mathbb{L}_i$ is the line bundle on $\overline{\mathcal{M}}_{0,n+3}$ such that over a moduli point $(C,x_1,\ldots,x_{n+3})$ the fiber is the cotangent space to C at x_i, $T^*_{x_i}C$.

In the Kapranov's model given by ψ_{n+3} , denote by $\mathcal{S}$ the collection of $n+2$ points in general position in $\mathbb{P}^n$ obtained by contraction of sections σ_i of the forgetful morphism of the $n+3$ marked point. Denote by S the set of indices parametrizing $\mathcal{S}$.

Further, denote by $\mathcal{X}_{n+2,(n-2)}$ the iterated blow-up of $\mathbb{P}^n$ along the strict transforms of *all* linear subspaces L_I of dimension at most $n-2$ spanned by sets of points $I\subset\mathcal{S}$ with $|I|\leq n-1$, ordered by increasing dimension. Notice that, in the notation of Section 1.3, for an effective divisor D, the iterated blown-up space $X_{n+2,(n-2)}$ along the linear subspaces that are in the base locus of $|D|$ is a resolution of singularities of D (see Corollary 4.10), so it depends on the divisor we start with. However, $\mathcal{X}_{n+2,(n-2)}$ depends only on the original set of $n+2$ points in $\mathbb{P}^n$.

In [34] Kapranov identifies the moduli space $\overline{\mathcal{M}}_{0,n+3}$ with the projective variety $\mathcal{X}_{n+2,(n-2)}$, in the notation of Section 1, by constructing birational maps from $\overline{\mathcal{M}}_{0,n+3}$ to $\mathbb{P}^n$ induced by the divisors ψ_i, for any choice of i with $1\leq i\leq n+3$.

We recall that the Picard group of $\mathcal{X}_{n+2,(n-2)}$ is spanned by a general hyperplane class and exceptional divisors, $\operatorname{Pic}(\mathcal{X}_{n+2,(n-2)})=\langle H,E_J\rangle$, where J is any nonempty subset of S with $1\leq|J|\leq n-1$.

Remark 6.1. In the Kapranov's model given by ψ_{n+3}, $\mathcal{X}_{n+2,(n-2)}$, there are $n+2$ Cremona transformations that are based on any subset of $n+1$ points of S, $S_i:=S\setminus\{p_i\}$. The ψ_i classes with $i\neq n+3$ correspond to the image of the Cremona transformation of a general hyperplane class H, based on the set S_i, denoted by $\operatorname{Cr}_i(H)=\operatorname{Cr}(H)$ in (4.10), while ψ_{n+3} corresponds to H.

Furthermore, we have the following identification:

$$\Delta_{\mathcal{I}} = \begin{cases} E_J, & |\mathcal{I}| \leq n, \\ H_J, & |\mathcal{I}| = n+1. \end{cases} \tag{6.1}$$

where E_J is the strict transform of the exceptional divisor obtained by blowing-up the linear cycle spanned points of J, while H_J is the strict transform of the hyperplane passing thought the points of J, namely

$$H_J := H - \sum_{\substack{I \subset J: \\ 1 \leq |I| \leq n-2}} E_I.$$

The F-*curves on* $\overline{\mathcal{M}}_{0,n+3}$ correspond to partitions of the index set

$$\mathcal{I}_1 \sqcup \mathcal{I}_2 \sqcup \mathcal{I}_3 \sqcup \mathcal{I}_4 = \{1, \ldots, n+3\}.$$

We remark that by definition, all subsets $\mathcal{I}_i$ are non-empty. We denote by $F_{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3,\mathcal{I}_4}$ the class of the corresponding F-curve. We have the following intersection table (see [36]).

$$F_{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3,\mathcal{I}_4} \cdot \Delta_{\mathcal{I}} = \begin{cases} 1 & \mathcal{I} = \mathcal{I}_i \sqcup \mathcal{I}_j, \text{ for some } i \neq j, \\ -1 & \mathcal{I} = \mathcal{I}_i, \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

We first describe the F-conjecture in a Kapranov's model using the coordinates of the Néron-Severi group $N^1(\mathcal{X}_{n+2,(n-2)})$. Consider a general divisor on $\mathcal{X}_{n+2,(n-2)}$ of the form

$$dH - \sum_{\substack{I \subset S: \\ 1 \leq |I| \leq n-1}} m_I E_I. \tag{6.3}$$

For a non-empty subset I of the points parametrized by S we define

$$a_I := \begin{cases} 0 & |I| \geq n, \\ 1 & |I| \leq n-1. \end{cases} \tag{6.4}$$

For any partition of the set of $n+2$ points $G \sqcup J \sqcup L = S$, set

$$\begin{aligned} A_{G,J,L} :&= d - a_G \cdot m_G - a_J \cdot m_J - a_L \cdot m_L + a_{J \sqcup L} \cdot m_{J \sqcup L} \\ &+ a_{J \sqcup G} \cdot m_{J \sqcup G} + a_{L \sqcup G} \cdot m_{L \sqcup G}. \end{aligned}$$

Example 6.2. Consider $|G| = n$ then J and L consist of one element each, say j and respectively l. Then $A_{G,J,L}$ is independent of G since $A_{G,J,L} = d - m_j - m_l + m_{jl}$ for $n \geq 3$. Whenever $|G| \leq n-1$ then $A_{G,J,L}$ depends on all three subsets.

Moreover, for any two non-empty subsets of S, I and J, set

$$b_{I \sqcup J} := \begin{cases} 0 & |I| + |J| \geq n, \\ 1 & |I| + |J| \leq n-1. \end{cases} \tag{6.5}$$

For any partition $I \sqcup G \sqcup J \sqcup L = S$ set $b_{I \sqcup J \sqcup L} := b_{I \sqcup (J \sqcup L)}$, as defined in (6.5), and

$$\begin{aligned} B_{I,G,J,L} :&= m_I - b_{I \sqcup G} \cdot m_{I \sqcup G} - b_{I \sqcup J} \cdot m_{I \sqcup J} - b_{I \sqcup L} \cdot m_{I \sqcup L} + \\ &+ b_{I \sqcup J \sqcup L} \cdot m_{I \sqcup J \sqcup L} + b_{I \sqcup G \sqcup J} \cdot m_{I \sqcup G \sqcup J} + b_{I \sqcup G \sqcup L} \cdot m_{I \sqcup G \sqcup L}. \end{aligned}$$

Example 6.3. If $|I| + |G| = n$ then J and L consist each of one element, call j and l respectively. In this case

- If $|I| = n-1$ then the subsets G, J and L consist of one element each and $B_{I,G,J,L} = m_I$.
- If $|I| = n-2$ then $B_{I,G,J,L} = m_I - m_{I\sqcup\{j\}} - m_{I\sqcup\{l\}}$.
- If $|I| \leq n-3$ then $B_{I,G,J,L} = m_I - m_{I\sqcup\{j\}} - m_{I\sqcup\{l\}} + m_{I\sqcup\{j\}\sqcup\{l\}}$.

Whenever $|I|+|G| \leq n-1$ then $B_{I,G,J,L}$ depends on the four subsets of the partition.

Remark 6.4. The number $A_{G,J,L}$ represents the intersection product between the divisor D and the corresponding F-curve contained in a hyperplane divisor and $B_{I,G,J,L}$ represents the intersection product between the divisor D and the F-curve contained in some exceptional divisor E_I.

Using the identification of boundary divisors (6.1) and the intersection table (6.2), it is easy to see that the following remark holds.

Remark 6.5 (The cone of F-nef divisors). A divisor on $\mathcal{X}_{n+2,(n-2)}$ of the form (6.3) is *F-nef* if the following sets of inequalities hold:

$$\begin{aligned} A_{G,J,L} &\geq 0, \quad \text{for any partition } G \sqcup J \sqcup L = S, \\ B_{I,G,J,L} &\geq 0, \quad \text{for any partition } I \sqcup G \sqcup J \sqcup L = S. \end{aligned} \tag{6.6}$$

This cone is often referred to as the *Faber cone* in the literature, see e.g. [28].

Conjecture 6.6 (F-conjecture). *A divisor on $\mathcal{X}_{n+2,(n-2)}$ of the form (6.3) is nef if and only if (6.6) holds.*

Take a general divisor with degree and multiplicities labeled as in (6.3). We will now describe general properties of the F-nef divisors in a Kapranov's model that are useful in computations.

Lemma 6.7. *Any F-nef divisor satisfies $d \geq m_I \geq 0$, for every $I \subset S$, and $m_I \geq m_J$, for every $I, J \subset S$ with $I \subset J$.*

Proof. We claim that these inequalities follow from (6.6). For $n = 2$ the claim is obvious, hence we assume $n \geq 3$. In fact, the following inequalities hold:

(1) $m_I \geq 0$, for every non-empty set I with $|I| = n-1$,
(2) $m_I \geq m_J$, for every non-empty sets I, J with $I \subset J$.

Claim (1) follows from Example 6.3 and Remark 6.5. To prove claim (2) we apply induction on $|I|$. For $i \neq j$ and $i, j \notin I$ we introduce the following notations: $I_i := I \sqcup \{i\}$ and $I_{ij} := I \sqcup \{i, j\}$. For the first step of induction consider the sets I and G with $|I| = n-2$ and $|G| = 2$. For any $i \neq j$ one has, by (6.6), that

$$m_I - m_{I_i} - m_{I_j} \geq 0.$$

Therefore claim (2) follows from claim (1) for any I with $|I| = n-2$. If $|I| \leq n-3$ the claim follows using backward induction on $|I|$. Indeed, by Example 6.3 we have

$$m_I - m_{I_i} - m_{I_j} + m_{I_{ij}} \geq 0,$$

therefore

$$m_I - m_{I_i} \geq m_{I_j} - m_{I_{ij}} \geq 0.$$

Since $|I_i| = |I| + 1$ and $I_i \subset I_{ij}$, the induction hypothesis holds for I_i, so the claim follows.

To see that $d \geq m_I$ we use Example 6.2 and claim (2) to obtain

$$d \geq m_i + (m_j - m_{ij}) \geq m_I.$$

□

6.2. **The F-conjecture holds for strict transforms on $\mathcal{X}_{n+2,(n-2)}$.** The main result of this section is Theorem 6.8, stating that the F-conjecture holds for all divisors on $\mathcal{X}_{n+2,(n-2)}$ that are strict transforms of an effective divisor on $X_{n+2,(0)}$.

In the notation of Section 1, D be any effective divisor on $X_{n+2,(0)}$ and let $\tilde{D}$ denote its strict transform on $X_{n+2,(n-2)}$. A general divisor on $\mathcal{X}_{n+2,(n-2)}$, for arbitrary coefficients d and m_I, is of the form (6.3) while the divisors $\tilde{D}$ have arbitrary coefficients d and m_i while $m_I := k_I$ defined in (1.4) for any index I with $|I| \geq 2$. We can consider $\tilde{D}$ a divisor on $\mathcal{X}_{n+2,(n-2)}$.

Theorem 6.8. *Assume $m_i \geq 0$, for all $i \in S$, then Conjecture 6.6 holds for $\tilde{D}$ on $\mathcal{X}_{n+2,(n-2)}$.*

Proof. To prove the claim, notice first that the effectivity of D implies $\sum_{i\in S} m_i \leq nd$ and $\sum_{i\in I} m_i \leq nd$, for all $I \subset S$ such that $|I| = n+1$. Moreover, since $m_i \geq 0$, we conclude by Theorem 4.1 and Remark 4.3.

We proved that $\tilde{D}$ is globally generated. Therefore $\tilde{D}$ is nef and in particular F-nef. □

Corollary 6.9. *For divisors of the form $\tilde{D} \geq 0$ on $\mathcal{X}_{n+2,(n-2)}$, the three properties of being F-nef, nef and globally generated are equivalent.*

Remark 6.10. The divisors in (6.3) for which $m_I < k_I$ are not nef, as they intersect negatively the class of a general line on the exceptional divisor E_I, for $2 \leq |I| \leq n-1$.

The divisors $\tilde{D}$ with $k_I \geq 1$, for some set I with $|I| \geq 2$ are not globally generated since they contract the exceptional divisors E_I.

Remark 6.11. Studying divisors interpolating higher dimensional linear cycles, L_I for $|I| \geq 2$, $m_I > k_I$, is a possible approach to the F-conjecture. Indeed, once the vanishing theorems are established by techniques developed in [12] and [24] they could be used for describing globally generated divisors or ample and nef cones of $\overline{\mathcal{M}}_{0,n}$. The description of ample divisors on $\overline{\mathcal{M}}_{g,n}$ is an important question originally asked by Mumford and conjectured by Fulton for $g = 0$.

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